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Numerical Solution of Dynamic Quantile Models[☆]

Luciano de Castro[a], Antonio F. Galvao[b,*], Andre Muchon[c]

[a] Department of Economics, University of Iowa, United States
[b] Department of Economics, Michigan State University, United States
[c] Möbius Capital and IMPA, United States

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ABSTRACT

This paper studies dynamic programming for quantile preference models, in which the agent maximizes the stream of the future τ-quantile utilities, for $\tau \in (0, 1)$. We suggest numerical methods, based on value function iterations, for solving the quantile recursive dynamic programming, and computing value and policy functions. In addition, we extend theoretical results to allow the dynamic quantile model to have a finite-horizon, instead of infinite-horizon. To illustrate the methods, we use an intertemporal consumption quantile model that has an explicit closed form solution for both the value and policy functions. Based on this example, we assess the accuracy of the numerical methods by computing and comparing theoretical and numerical value and policy functions, for several combinations of the parameters – discount factor, elasticity of intertemporal substitution, and risk attitude, which is measured by the quantile. Results document evidence that the suggested algorithm provides numerical solutions that are very close to theoretical counterparts, and also illustrate the usefulness and practicality of the proposed methods.

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1. Introduction

The dynamic programming framework has been extensively used in economic modeling because it is sufficiently rich to deal with most problems involving sequential decisions over time and under uncertainty.[1] Recently, de Castro and Galvao (2019) proposed a new infinite-horizon dynamic model for an economic agent, who, when selecting among uncertain alternatives, chooses the one with the highest τ-quantile of the stream of future utilities for a fixed $\tau \in (0, 1)$, instead of the standard expected utility. This quantile preference model is tractable, simple to interpret, and allows for separation between risk aversion and elasticity of intertemporal substitution. Moreover, some quantile dynamic models have closed form solutions which are not available for their expectation counterpart.[2] The quantile model also possesses all the de-

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* Corresponding author.
E-mail address: agalvao@msu.edu (A.F. Galvao).

[1] There is an extensive literature on dynamic nonlinear rational expectation models. The great majority of models of dynamic maximization employ the expected utility and have been workhorses in several economic fields. We refer the reader to more comprehensive works, such as Stokey et al. (1989) and Ljungqvist and Sargent (2012). Another related segment of the literature studies recursive utilities. We refer the reader to Epstein and Zin (1989), Marinacci and Montrucchio (2010), Bommier et al. (2017), among others.

[2] Rostek (2010) discusses several advantages of the static quantile preference, such as robustness, ability to deal with categorical (instead of continuous) variables, and the flexibility of offering a family of preferences indexed by quantiles.

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sirable standard properties of recursive models. In particular, it is dynamically consistent, and the principle of optimality holds. Moreover, the corresponding dynamic problem yields a value function, via a fixed point argument, the value function is monotone, concave, and differentiable, and it is also possible to derive the corresponding Euler equation. There is a growing literature on economic models using quantile preferences, see, e.g., among others, Manski (1988), Chambers (2007, 2009), Bhattacharya (2009), Rostek (2010), Giovannetti (2013), Long et al. (2021), Baruník and Čech (2021), He et al. (2021), de Castro et al. (2022a), and Baruník and Nevrla (2022).

Solving dynamic stochastic optimization problems numerically is very important in practice. There is a large literature discussing numerical solutions for dynamic economic models under uncertainty. For reviews of methods for numerically solving intertemporal economic models, see, among others, Taylor and Uhlig (1990), Rust (1996, 2008), Gaspar and Judd (1997), Judd (1998), Marimon and Scott (1999), Santos (1999), Christiano and Fisher (2000), Miranda and Fackler (2002), Adda and Cooper (2003), Aruoba et al. (2006), Stachursky (2009), Den Haan (2010), Kollmann et al. (2011), Ljungqvist and Sargent (2012), Miao (2013), and Maliar and Maliar (2014). Most of the available existing methods work with the expected utility case. Nevertheless, quantiles are nonlinear operators, which precludes the use of some simplifications that are only valid for expectations. Recently, numerical computation using recursive preferences together with non-linearity has received attention – see, e.g., Caldara et al. (2012) and van Binsbergen et al. (2012) for applications of Epstein and Zin (1989, 1991) preferences to DSGE models. Solving these models numerically may be a challenge.

This paper extends the literature in two fronts. First, we extend existing theoretical results on the intertemporal quantile model in an important dimension that is useful to applied work. We allow the model to have a finite-horizon instead of being restricted to infinite-horizons. We show that, in the finite case, the sequence of corresponding value-functions is well-defined. Finite-horizon problems are often encountered in making life-cycle planning decisions on optimal consumption, savings, portfolio choice, etc. This is a significant extension since it allows for applications of many more realistic models.

Second, we suggest numerical methods for solving the dynamic stochastic programming for quantile recursive models based on value function iterations. The numerical solution provides a link between the theory for quantile dynamic programming and empirical analyses of dynamic optimization problems. The need for numerical tools arises from the fact that, in general, dynamic programming problems do not possess tractable closed form solutions. Hence, numerical techniques must be used to approximate their solutions. In particular, we suggest a procedure based on value function iterations where the main difference with standard methods is that one is required to compute the conditional τ-quantile of the value function, instead of the conditional average. We note that, for the standard conditional expectation case, given a particular state and conditional probabilities in the transition matrix, the conditional average is easily obtained by calculating the sample average over the states using the corresponding probabilities. For the quantile model, the calculation of the conditional τ-quantile is also simple. Given the conditional probabilities for the current shock, the state, and the choice, one orders all the possible future value function outcomes, and sums the corresponding probabilities from the lowest value to the highest, until the sum of probabilities is at least the given $\tau \in (0, 1)$.[3]

Numerical methods are provided for both the finite and infinity horizon cases. Both algorithms use the Bellman equation to compute the value function. For the former case, backward iterations on an initial guess are employed. In the latter case, the value function is iterated until convergence occurs. In this paper, we focus on the case that the shock is discrete with a Markov transition matrix. Nevertheless, we provide a brief discussion on an extension of the algorithm to the continuous shock case using the Tauchen (1986) finite state Markov-chain approximation.

We provide results assessing the accuracy of the proposed numerical methods and illustrating its practicality. To do so, we use a simple intertemporal consumption model where the economic agent is characterized by quantile preferences and decides on the intertemporal consumption and savings (assets to hold) over time, subject to a linear budget constraint. This model is characterized by three parameters — discount factor, elasticity of intertemporal substitution (EIS), and risk attitude. Notice that in quantile models, the risk attitude is captured by the quantile τ, while the other two parameters play a similar role to those in dynamic expected utility models. However, in the standard expected utility, the EIS is indissociable from risk aversion.[4] de Castro et al. (2022b) derive explicit algebraic closed form solutions for the value and policy functions for this intertemporal consumption model.[5] Hence, we are able to calculate these functions both theoretically and numerically, for different value of the parameters. We are then able to evaluate the performance of the numerical procedures by comparing the results for the numerical computation with the corresponding closed form solutions.

To compute the intertemporal consumption quantile model, numerically and theoretically, we specify an isoelastic utility function and calculate the value and asset allocation policy functions for several combinations of the parameters, as well as different probabilities in the transition matrices, using independent and identically distributed (iid) and Markov shocks. Results from these exercises show evidence that the suggested numerical methods to solve the dynamic quantile model approximate the theoretical solutions very closely, and provide a high degree of accuracy. Hence, researchers applying this solution algorithm can be confident that their quantitative answers are sound.

[3] Notice that the computation of the conditional quantile here is different from quantile regression methods. We refer the reader to Koenker (2005) and Koenker et al. (2018) for reviews of quantile regression methods.

[4] It has been well documented in the literature that it is not possible to separate the intertemporal substitution from the risk attitude parameters when using standard dynamic models based on the EU (see, e.g., Hall, 1978; 1988).

[5] Existence of closed form expressions for the intertemporal consumption framework is an advantage of the quantile model. Nevertheless, we note that not all quantile dynamic models have closed form solutions. Thus, the suggested numerical algorithm is an useful tool for more general models.

To illustrate the usefulness and practicality of the proposed methods, we compare numerical results for the quantile model with the standard expected utility case. We use different parametrizations that change probabilities of the shock, the risk aversion, and the EIS. For each design, we compute the asset allocation decision rule and the value function. These exercises illustrate several interesting features of the dynamic quantile model. First, we verify that both value and policy functions are monotone increasing. Second, results illustrate the flexibility and practicality of the quantile framework. The quantile model is characterized by three parameters, while the standard expected utility is only characterized by two parameters. Thus, in the quantile case, for a given discount factor, asset allocation varies with both risk attitude and EIS. When EIS > 1, asset allocation is relatively smaller for a more risk averse agent ($\tau = 0.25$) compared to a less risk averse ($\tau = 0.75$). On the other hand, for EIS < 1, the volume of assets bought for the next period (savings) is relatively larger for a more risk averse agent ($\tau = 0.25$) compared to a less risk averse ($\tau = 0.75$). As EIS decreases, the responsiveness of the growth rate of consumption to the real interest rate decreases. If consumption is less responsive to changes, savings is more responsive. Thus, the asset allocation increases for all quantiles, but it increases relatively more for a more risk averse agent. This illustrates the advantage of being able to separate risk and EIS, and hence evaluate effects of changing EIS on next period asset allocation decisions, while maintaining risk attitudes constant.

Finally, we simulate the intertemporal consumption model. Applied economists often characterize the behavior of the model through statistics from simulated paths of the economy. We simulate the model for 10,000 periods and compute density functions of the allocation of asset savings for the next period for different combinations of parameters. Results show large heterogeneity in the distributions across different risk attitudes and EIS.

The remainder of the paper is structured as follows. Section 2 reviews the dynamic quantile model. Section 3 presents the general infinite-horizon model, and its numerical algorithm for computation. Section 4 studies the finite-horizon quantile model, and its corresponding numerical algorithm for computation. Section 5 provides numerical results using the intertemporal consumption model as an example. We assess the numerical performance of the computational algorithms, compare results with the standard expected utility model, and provide simulation results. Finally, Section 6 concludes.

2. The recursive quantile model

This section describes the dynamic quantile model introduced by de Castro and Galvao (2019), whose numerical implementation is the main object of the current paper. In Section 2.2 below, we specialize this model to a simple standard consumption-based economy model in order to illustrate some features of the general model. Moreover, later in the paper, we return to this example to assess the performance of the numerical methods as well as to exemplify the proposed methods. The general quantile model is further analyzed in Section 3.

2.1. The quantile model

Let $x_t \in \mathcal{X}$ and $z_t \in \mathcal{Z}$ denote, respectively, the state and the shock in period t, both of which are known by the decision maker (DM) at the beginning of period t, where $\mathcal{X}$ is the set of possible states and $\mathcal{Z}$ is the set of shocks.

For simplicity, we assume that the set of shocks is finite, that is, $\mathcal{Z} \equiv \{s^1, \ldots, s^k\}$.[6] The random shocks will follow a time-invariant (stationary) Markov process, defined by the stochastic matrix $P = \left(p_{ij}\right)_{k\times k}$, that is, the probability of moving from state s^i to state s^j is $\Pr(s^j|s^i) = p_{ij}$.

Let $y_t \in \mathcal{Y}$ denote the choice that the agent makes at time t, where the choice set $\mathcal{Y}$ can finite or infinite. Given the state x_t and the shock z_t, the DM chooses $y_t \in \Gamma(x_t, z_t) \subseteq \mathcal{Y} \subseteq \mathbb{R}^q$, where $\Gamma : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{Y}$ is the budget set correspondence.

The period t utility is given by $u(x_t, y_t, z_t)$, where $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ is a utility function. Given current period state x_t, choice y_t and the next period's shock z_{t+1}, the next state x_{t+1} is defined by the transition function $\sigma : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$, that is,

$$x_{t+1} = \sigma(x_t, y_t, z_{t+1}). \tag{1}$$

In other words, this function determines the next period state variable x_{t+1} as function of the current state x_t, the choice y_t, and the shock z_{t+1} realized at the beginning of period $t+1$. The simplest case is, of course, when the decision maker chooses directly the next period state, that is, $y_t = x_{t+1}$ or $x_{t+1} = \sigma(x_t, y_t, z_{t+1}) = y_t$. A natural example of this case occurs if x_t is the amount of asset holdings at the beginning of period t and y_t is how much asset is bought in period t and held for period $t+1$. Stokey et al. (1989, p. 239–240) uses the following example to illustrate cases in which the next state is influenced by the shock and a more elaborate transition function σ is required.

Example 1. There is one good (corn), which can be consumed or stored to be used as seed in the next period. The state in period t is the amount of corn x_t that is available to be planted. Once planted, this seed yields $f(x_t)$, where f is a production function. The choice in period t is how much to store for next period, y_t. Thus, the consumption is $c_t = f(x_t) - y_t$. There is

[6] Dynamic quantile preferences can be modeled with an abstract set of shocks, in which $\mathcal{Z}$ is only required to be a metric space, as de Castro et al. (2022b) show. However, in this general setting some measurability and regularity conditions for the conditional probabilities are required. Since the focus of this paper is on numerical methods, which require discretization in any case, we avoid these technicalities and work directly with the discrete case. We briefly discuss an extension to continuous shocks in Remark 1.

a population of mice in the storehouse that may consume part of the corn, but whose size is stochastic. Assume that the effect of mice is additive, that is, the next period state x_{t+1} will be the amount stored y_t, minus the amount affected by the mice, z_{t+1}. In this case, we have:

$$\begin{aligned} x_{t+1} &= \sigma(x_t, y_t, z_{t+1}) = y_t - z_{t+1}; \\ u(x_t, y_t, z_t) &= U[f(x_t) - y_t] \\ \Gamma(x_t, z_t) &= [0, f(x_t)]. \end{aligned}$$

In order to complete the definition of the dynamic quantile preference model, we need to define quantiles. Recall that if the cumulative distribution function (c.d.f.) F_X of a random variable X is invertible, then its τ-quantile is simply the preimage of $\tau \in (0,1)$, that is, $Q_\tau[X] = F_X^{-1}(\tau)$. Since we are working with discrete shocks, the c.d.f. will not be invertible and we can define the quantile as a number between $\sup\{x \in \mathbb{R} : F_X(x) \leqslant \tau\}$ and $\inf\{x \in \mathbb{R} : F_X(x) \geqslant \tau\}$. Following the usual convention (see Koenker (2005), for instance), we will adopt the second definition. Therefore, the τ-conditional quantile of a random variable X given current shock z_t, for $\tau \in (0,1)$ is defined as: $Q_\tau[X|z_t] \equiv \inf\{x \in \mathbb{R} : \Pr[X \leqslant x|z_t] \geqslant \tau\}$. We will use this definition in the case that $X = g(z_{t+1})$ for a function $g : \mathcal{Z} \to \mathbb{R}$ of next period shock z_{t+1}. In this case, the probability $\Pr[X \leqslant x|z_t]$ is just the sum $\sum_{j \in A_x} p_{ij}$, for $A_x \equiv \{j \in \{1, \ldots, k\} : g(s^j) \leqslant x\}$, provided that $z_t = s^i$.

In the general dynamic quantile model, the intertemporal choices can be represented by the maximization of a value function $V : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ that satisfies the recursive equation:

$$V(x_t, z_t) = \max_{\substack{y_t \in \Gamma(x_t, z_t) \\ x_{t+1} = \sigma(x_t, y_t, z_{t+1})}} \Big\{ u(x_t, y_t, z_t) + \beta Q_\tau[V(x_{t+1}, z_{t+1}) \,|\, z_t] \Big\}. \tag{2}$$

In this problem, $Q_\tau[\cdot \mid \cdot]$ is the τ-conditional quantile function for $\tau \in (0,1)$, $V(\cdot,\cdot)$ is the value-function, the quantile-τ is given, $\beta \in (0,1)$ is the discount factor and $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ is the utility function. The information set at time t is simply z_t.

Now we state the main assumptions used for establishing the properties of the model.

Assumption 1. The following properties are maintained throughout the paper:

(i) $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are metric spaces;
(ii) $\mathcal{Z}$ is either connected or finite and the $\{z_t\}$ follows a Markov process with transition matrix $P = (p_{ij})_{k \times k}$ or fixed p.d.f. $f(z_{t+1}|z_t)$;
(iii) $\sigma : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}_+$ is continuous;
(iv) $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ is continuous and bounded;
(v) The correspondence $\Gamma : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{Y}$ is continuous, with nonempty, compact and convex values.

The above conditions are all standard in the literature, see e.g. Stokey et al. (1989). Assumption 1(i) allows the state, choice, and shock sets to me metric spaces. Assumption 1(ii) allows for the shocks to be either discrete or continuous. The restrictions in Assumptions 1(iv) and (v) on the utility and budget functions, respectively, are standard in the literature. Finally, Assumption 1(iii) restricts the transition function in equation (1) to be continuous and bounded. This is a weak condition, and is in general satisfied in standard model, as we further discuss and illustrate below. Indeed, in the next section, we introduce an intertemporal consumption model with quantile preferences that illustrates the usefulness and flexibility of this framework. In particular, it shows how Assumption 1 is naturally satisfied.

2.2. Intertemporal consumption

We illustrate the above model with the intertemporal consumption dynamic model analyzed by de Castro et al. (2022b). This model adapts the standard intertemporal consumption and life-cycle analysis introduced in Modigliani and Brumberg (1954) to the dynamic quantile preference. This framework is particularly interesting, since it is possible to show that it leads to closed form solutions, which allow us to compare analytical solutions with numerical results obtained from the proposed methods. This is implemented in Section 5.

The state x_t is the amount invested in a risky asset in period $t-1$ and z_t is the return that this asset yields in period t. Thus, at the beginning of period t, the DM has $x_t z_t$ to consume or save for the next period. If the DM saves x_{t+1}, the consumption is $x_t z_t - x_{t+1}$. In this case, we have:

$$\begin{aligned} x_{t+1} &= \sigma(x_t, y_t, z_{t+1}) = y_t; \\ u(x_t, y_t, z_t) &= U(x_t z_t - y_t) \\ \Gamma(x_t, z_t) &= [0, x_t z_t]. \end{aligned}$$

Thus, we can substitute y_t for x_{t+1} and write the recursive equation (2) simply as:

$$V(x_t, z_t) = \max_{x_{t+1} \in [0, x_t z_t]} \Big\{ U(x_t z_t - x_{t+1}) + \beta Q_\tau[V(x_{t+1}, z_{t+1}) \,|\, z_t] \Big\}.$$

We discuss the economic interpretation of this model in more detail in Section 5.3 below. Notice that Assumption 1 is satisfied, provided that the utility function U is continuous.

2.3. Remarks on the dynamic quantile model

The general theoretical properties of the model in (2) are established in de Castro et al. (2022b); see also de Castro and Galvao (2019). They show that the quantile model possesses all the desirable standard properties of recursive models. In particular, the model is dynamically consistent, and the principle of optimality holds. The optimization problem leads to a contraction, which therefore has a unique fixed point. This fixed point is the value function of the problem and satisfies the Bellman equation. They also prove that the value function is concave and differentiable, thus establishing the quantile analog of the envelope theorem. Additionally, they derive the corresponding Euler equation.

The interpretation of the recursive problem in (2) is very similar to the standard expected utility model. The value function at time t is equal to the utility of consumption at time t plus the discounted value of the τ-*quantile* – instead of the expectation – of the value function at time $t+1$.

For detailed discussions on axiomatization and risk attitude of the dynamic preferences see de Castro and Galvao (2022). The risk attitude of a quantile maximizer is captured by τ in static models (see, e.g., Manski (1988) and Rostek (2010)). Using the notion of quantile-preserving spread introduced by Mendelson (1987), de Castro and Galvao (2022) adapt the definition of risk for dynamic models in Epstein and Zin (1989) to show that the single dimensional parameter $\tau \in (0,1)$ captures the risk attitude in dynamic quantile models. Hence, this model admits a notion of comparative risk attitude, where an agent with quantile given by τ_1 is more risk preferring than another agent with quantile given by τ_2 if $\tau_1 > \tau_2$, independently of the functional form of the utility function.

An important property of the dynamic quantile model above is that it allows for separation between risk aversion and elasticity of intertemporal substitution (EIS). The EIS is determined by the concavity of the utility function, exactly as in a model without risk or uncertainty. The concavity of the utility function has no bearing on the risk attitude of a quantile maximizer because single period quantile preferences are invariant to the utility function. For more details, see discussion on de Castro and Galvao (2019) or de Castro and Galvao (2022).

3. Infinite-Horizon quantile model and computation

This section considers a general quantile dynamic model for infinity-horizon and provides numerical methods using value function iterations to solve the model numerically.

3.1. Quantile value function representation

We start by establishing the existence of the value function for the infinite-horizon problem. Let $\mathcal{C}$ denote the set of functions $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$. Since $\mathcal{X}$ and $\mathcal{Z}$ are finite, these functions are necessarily continuous. We endow the set $\mathcal{C}$ with the supnorm, that is, for any $v \in \mathcal{C}$, we define

$$\|v\| = \sup_{(x,z)\in\mathcal{X}\times\mathcal{Z}} |v(x,z)|. \tag{3}$$

Endowed with this norm, $\mathcal{C}$ is a Banach space.[7]

We want to find a function $v_\tau \in \mathcal{C}$ that represents the continuation utility value for the decision maker (DM) that maximizes a τ-quantile. That is, given state $x \in \mathcal{X}$ and shock $z \in \mathcal{Z}$, $v_\tau(x,z)$ represents the utility value that the DM attributes to the initial point (x,z). Let $w \in \mathcal{Z}$ denote the shock next period. Notice that in this case, the DM will choose $y \in \Gamma(x,z)$ and the next period state will be $x' = \sigma(x,y,w)$, when the next period shock w is realized. The value $v_\tau(x',w)$ will represent the value in the next period. Since y is chosen optimally, the value function has to satisfy the following consistency equation:

$$\begin{aligned} v_\tau(x,z) &= \sup_{y\in\Gamma(x,z)} \left\{ u(x,y,z) + \beta \mathrm{Q}_\tau[v_\tau(x',w)|z] \right\} \\ &= u(x,y^*,z) + \beta \mathrm{Q}_\tau[v_\tau(\sigma(x,y^*,w),w)|z], \end{aligned} \tag{4}$$

where $y^* \in \Gamma(x,z)$ is the optimal choice, that realizes the supremum on the right. Notice that the only difference with respect to the standard expected utility problem is the use of the τ-quantile operator $\mathrm{Q}_\tau[\cdot]$ in place of the expectation operator $\mathrm{E}[\cdot]$. Despite this difference, we can use the standard method to find the value function v that satisfies (4), as we show now.

As usual, we have to define a transformation $\mathbb{T}_\tau : \mathcal{C} \to \mathcal{C}$, whose fixed point will be the value function satisfying (4). Indeed, given $v \in \mathcal{C}$, we define a function $\mathbb{T}_\tau(v) \in \mathcal{C}$ by defining its value in each $(x,z) \in \mathcal{X}\times\mathcal{Z}$ through the following equation:

$$\mathbb{T}_\tau(v)(x,z) \equiv \sup_{y\in\Gamma(x,z)} \left\{ u(x,y,z) + \beta \mathrm{Q}_\tau[v(\sigma(x,y,w),w)|z] \right\}. \tag{5}$$

Fortunately, as in the expected utility case, this transformation is a contraction and has a fixed point. This is established by the following:

[7] In fact, since $|\mathcal{X}| = p$ and $|\mathcal{Z}| = k$, $\mathcal{C}$ can be identified with $\mathbb{R}^{p\cdot k}$.

Lemma 1. *Under Assumption 1, $\mathbb{T}_\tau$ is a contraction with modulus β, that is, $\forall v, v' \in \mathcal{C}$*

$$\|\mathbb{T}_\tau(v) - \mathbb{T}_\tau(v')\| \leqslant \beta\|v - v'\|. \tag{6}$$

Therefore, $\mathbb{T}_\tau$ has a unique fixed-point $v_\tau \in \mathcal{C}$, that is, $\mathbb{T}_\tau(v_\tau) = v_\tau$.

Proof. See Appendix A.1. □

Notice that the equation $\mathbb{T}_\tau(v_\tau) = v_\tau$ above is just another way of rewriting (4). Thus, the unique fixed point of the problem is the value function of the problem. Lemma 1 follows from de Castro et al. (2022b).

It is useful to illustrate how we can use the above result to find the value function. Let us say that we pick up any arbitrary value function $v \in \mathcal{C}$ (it can even be a constant function). Put $v^0 = v$ and define $v^1 = \mathbb{T}_\tau(v^0)$, that is, obtain the function defined from v^0 by using the operator $\mathbb{T}_\tau$. Then, repeat this procedure: assuming that we have obtained v^n, define $v^{n+1} = \mathbb{T}_\tau(v^n)$, that is,

$$v^{n+1}(x, z) \equiv \sup_{y \in \Gamma(x,z)} \Big\{ u(x, y, z) + \beta \mathrm{Q}_\tau[v^n(\sigma(x, y, w), w)|z] \Big\}. \tag{7}$$

This procedure will eventually lead to the value function. More formally, we have the following:

Proposition 1. *Under Assumption 1, for any $v \in \mathcal{C}$, define $v^0 = v$ and, for $n = 0, 1, 2, \ldots$, define recursively $v^{n+1} = \mathbb{T}_\tau(v^n)$ as in (7). Then, $v_\tau = \lim_{n\to\infty} v^n$, where v_τ is the unique fixed point of $\mathbb{T}_\tau$. Moreover,*

$$\|v^n - v_\tau\| \leqslant \beta^n \|v^0 - v_\tau\|.$$

Proof. See Appendix A.1. □

Proposition 1 follows from de Castro and Galvao (2019, Theorem 3.11, p. 1913). Next, we provide a numerical algorithm to compute a solution for this problem.

3.2. Algorithm to compute the value and policy functions

The traditional method of solving dynamic stochastic optimization problem with expected valuation is through "Value Function Iterations." This method is known to work with the Bellman equation to compute the value function iterations on an initial guess until convergence. While sometimes slower than competing methods, it is trustworthy in that it reflects the result, stated in Lemma 1 and Proposition 1, that under certain conditions, the solution of the Bellman equation can be reached by iterating the value function starting from an arbitrary initial value. The main different ingredient in our proposal is the computation of the conditional τ-quantile instead of the conditional average, in Step 3 below. This is simple to calculate in practice. Given conditional probabilities for the current shock, the state, and the choice, one orders all the possible future value function outcomes, and sums the corresponding probabilities from the lowest value to the highest, until the sum of probabilities is at least $\tau \in (0, 1)$.

First, we describe how to proceed to find a numerical estimate of the value function v_τ for the general model described above. This is given in Steps 1–5.

1. **Functional forms and parameters.** Choose the model and the functional form for the utility function $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$, the budget set $\Gamma : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y}$, the transition function $\sigma : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ and stochastic transition matrix P.
   (a) Choose the value of the parameters of the functional form. The parameters can be taken from a grid of possible values.
   (b) Choose the initial value function $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$. This can be a constant function $v(x, z) = 0$, for instance.
   (c) This leads to the function $g : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ defined by
$$g(x, y, z, w) \equiv u(x, y, z) + \beta v(\sigma(x, y, w), w). \tag{8}$$
2. **Discretize and initialize variables.** Represent the functions through matrices or arrays and initialize the variables of the algorithm.
   (a) Fix $\mathcal{X} = \{a^1, \ldots, a^p\}$ and $\mathcal{Z} = \{s^1, \ldots, s^k\}$.
   (b) Initialize the stochastic transition matrix $P = (P[i, j])_{k\times k}$ so that $P[i, j] = p_{ij}$ for each $(i, j) \in \{1, \ldots, k\}^2$.
   (c) Fix a discretization for $\mathcal{Y} = \{b^1, \ldots, b^q\}$.
   (d) Represent the function $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ by a $p \times q \times k$ array $U = (U[i_x, i_y, i_z])_{p\times q\times k}$, defined for each $(i_x, i_y, i_z) \in \{1, \ldots, p\} \times \{1, \ldots, q\} \times \{1, \ldots, k\}$,
$$U[i_x, i_y, i_z] \equiv u\big(a^{i_x}, b^{i_y}, s^{i_z}\big).$$
   (e) Represent the function $\sigma : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ by a $p \times q \times k$ array of integers $N = (S[i_x, i_y, i_w])_{p\times q\times k}$, defined for each $(i_x, i_y, i_w) \in \{1, \ldots, p\} \times \{1, \ldots, q\} \times \{1, \ldots, k\}$,
$$S[i_x, i_y, i_w] = j \in \{1, \ldots, p\} \quad \text{if} \quad a^j = \sigma\big(a^{i_x}, b^{i_y}, s^{i_w}\big).$$
   (f) Represent $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ by a $p \times k$ matrix $W = (W[i_x, i_z])_{p\times k}$, defined for each $(i_x, i_z) \in \{1, \ldots, p\} \times \{1, \ldots, k\}$,
$$W[i_x, i_z] \equiv v\big(a^{i_x}, s^{i_z}\big).$$

(g) Initialize the array $G = \left(G[i_x, i_y, i_z, i_w]\right)_{p\times q\times k\times k}$, defined for each $(i_x, i_y, i_z, i_w) \in \{1,\dots,p\}\times\{1,\dots,q\}\times\{1,\dots,k\}\times\{1,\dots,k\}$ by $G[i_x, i_y, i_z, i_w] = 0$.
(This records the value of the function g defined in (8).)
(h) Initialize the array $H = \left(H[i_x, i_y, i_z]\right)_{p\times q\times k}$, defined for each $(i_x, i_y, i_z) \in \{1,\dots,p\}\times\{1,\dots,q\}\times\{1,\dots,k\}$ by $H[i_x, i_y, i_z] = 0$. (This records the quantile of g in (8).)
(i) Compute the budget set $\Gamma(x, z)$ and record them as $B = (B[i_x, i_y, i_z])_{p\times q\times k}$ defined by:

$$B[i_x, i_y, i_z] = \begin{cases} 1, & \text{if } b^{i_y} \in \Gamma(a^{i_x}, s^{i_z}) \\ 0, & \text{otherwise} \end{cases}$$

(j) Initialize the array $Y = (Y[i_x, i_z])_{p\times k}$, defined for each $(i_x, i_z) \in \{1,\dots,p\}\times\{1,\dots,k\}$ by $Y[i_x, i_z] = 1$. (This records the optimal policy function.)

3. **Compute the conditional τ-quantile.** For each (x, y, z), obtain the τ-quantile of $g(x, y, z, w)$ with respect to w:
(a) Make the value function v equal to the candidate: $V = W$.
(b) Obtain the function $g : \mathcal{X}\times\mathcal{Y}\times\mathcal{Z}\times\mathcal{Z}\to\mathbb{R}$, that is, for each $(i_x, i_y, i_z, i_w) \in \{1,\dots,p\}\times\{1,\dots,q\}\times\{1,\dots,k\}\times\{1,\dots,k\}$ define

$$G[i_x, i_y, i_z, i_w] \equiv U[i_x, i_y, i_z] + \beta V[S[i_x, i_y, i_w], i_w].$$

(c) For each (i_x, i_y, i_z), order $G[i_x, i_y, i_z, \cdot]$ to obtain $i_w^{(1)}, i_w^{(2)}, \dots, i_w^{(k)} \in \{1,.,k\}$ so that

$$G[i_x, i_y, i_z, i_w^{(1)}] \leqslant G[i_x, i_y, i_z, i_w^{(2)}] \leqslant \ldots \leqslant G[i_x, i_y, i_z, i_w^{(k)}].$$

(d) For each (i_x, i_y, i_z), find the lowest $j \in \{1,.,k\}$ for which

$$P[i_z, i_w^{(1)}] + \cdots + P[i_z, i_w^{(j)}] \geqslant \tau.$$

(e) Define the array $H = \left(H[i_x, i_y, i_z]\right)_{p\times q\times k}$, defined for each $(i_x, i_y, i_z) \in \{1,\dots,p\}\times\{1,\dots,q\}\times\{1,\dots,k\}$ by

$$H[i_x, i_y, i_z] = G[i_x, i_y, i_z, i_w^{(j)}].$$

4. **Compute the maximum value function and save the optimal policy.** For each (x, z), find y that maximizes $Q_\tau[g(x, y, z, w)|z]$.
(a) For each (i_x, i_z), maximize $j \mapsto H[i_x, j, i_z]$, that is, find $i_y^* \in \{1,\dots,q\}$ such that

$$H[i_x, i_y^*, i_z] = \max_{1\leqslant j\leqslant q, B[i_x, j, i_z]=1} H[i_x, j, i_z].$$

(b) Define $Y[i_x, i_z] = i_y^*$.
5. **Check convergence.** Obtain the function $\mathbb{T}_\tau(v)$ (recorded as W) and test whether it is already a good approximation of v_τ
(a) For each (i_x, i_z), define $W[i_x, i_z] = H[i_x, i_y^*, i_z]$.
(b) Obtain the error

$$e \equiv \|W - V\| = \max_{(i_x, i_z)\in\{1,.,p\}\times\{1,\dots,k\}} \left|W[i_x, i_z] - V[i_x, i_z]\right|.$$

(c) Compare the error e with the maximum error $\epsilon > 0$: if $e > \epsilon$, repeat from item (3) above; otherwise, stop.

After we have obtained the final value function, we find a numerical estimate of the corresponding optimal policy function. That is, once v_τ is obtained, we can obtain, for each (x, z), the optimal policy y^* that maximizes $Q_\tau[g_t(x, y, z, w)|z]$. This can be done using the following simple algorithm:

(a) Initialize the array $Y = (Y[i_x, i_z])_{p\times k}$, defined for each $(t, i_x, i_z) \in \{1,\dots,p\}\times\{1,\dots,k\}$ by $Y[i_x, i_z] = 0$. (This records, after the value function is found, the optimal policy y_t^* at time t as a function of (x, z) defined by (i_x, i_z).)
(b) For each (i_x, i_z), order $H[i_x, \cdot i_z]$ to obtain i_y^* such that

$$H[i_x, i_y^*, i_z] = \max_{1\leqslant j\leqslant q, B[i_x, j, i_z]=1} H[i_x, j, i_z].$$

(c) Define $Y[i_x, i_z] = i_y^*$.

3.3. Simulation of the model

The last step is to simulate the path of the behavior and utility of the agent. For simplicity, we describe the algorithm for the simulation only in the standard notation of Section 2. Once we have obtained the value function $v(x, z)$ policy function $y^*(x, z)$, we can simulate the model as follows, assuming that all the previous values are known and fixed:

1. **Choose the initial values.** Fix the initial state (x_0, z_0), $y_0 = y^*(x_0, z_0)$, and the number of periods T in the simulation;
2. **Initialize values** Fix $t = 1$;
3. **While** $t \leqslant T$**,** do the following:

(a) Given z_{t-1}, obtain z_t using the stochastic transition matrix $P = (P[i, j])_{k\times k}$ and a random number generator.
(b) Obtain $x_t = \sigma(x_{t-1}, y_{t-1}, z_t)$;
(c) Obtain $y_t = y^*(x_t, z_t)$;
(d) Put $t = t + 1$.

We present the following remarks before closing this section.

Remark 1. If one is interested in continuous shocks, it is possible to follow the technique presented in Tauchen (1986) and discretize $\mathcal{Z}$ in I points $\{z_i\}_{i=1}^I$. In fact, one approximates an autoregressive process by a Markov chain. The method determines the optimal discrete points $\{z_i\}$ and the transition matrix $P = (P[i, j])_{k\times k} = Prob(z_t = z_{t-1}|z_{t-1} = z_j)$ such that the Markov chain mimics an AR(1) process. Naturally, the approximation is only good if I is large enough.

Remark 2. The previous algorithm is designed to accommodate general dynamic quantile models. In some specific situations, one may use the Euler equation and/or implied constrains by the model to accelerate convergence and improve computing time (see, e.g., Caldara et al. (2012, p.196)).

4. Finite-Horizon quantile model and computation

In this section, we first describe a general finite-horizon quantile dynamic model and show that the recursive model is well-defined. Second, we provide an algorithm that uses value function iterations to solve the model numerically.

4.1. The sequence of value functions

In order to introduce the stochastic model for finite-horizon, it is useful to recall first the deterministic finite time dynamic problem:

$$\sum_{t=1}^{T} \beta^t u(x_t, y_t, z_t), \tag{9}$$

where z_t is constant (nonrandom). This model is suitable for the resolution by backward induction. Suppose that we are in the last period T and need to choose y_T in order to maximize $u(x_T, y_T, z_T)$, that is, $v_T(x_t, z_t) \equiv \max_{y_t\in\Gamma(x_t,z_t)} u(x_t, y_t, z_t)$. In fact, we define recursively a sequence of value functions in the following manner. Put $v_{T+1}(x, z) = 0$ and for $t = T, T-1, \dots, 1$, we define recursively:

$$v_t(x_t, z_t) \equiv \max_{y_t\in\Gamma(x_t,z_t)} \Big\{u(x_t, y_t, z_t) + \beta v_{t+1}(x_{t+1}, z_{t+1})\Big\}, \tag{10}$$

where $x_{t+1} = \sigma(x_t, y_t, z_{t+1})$. Notice that the recursive definition (10) allows the choice to be made period by period. One can show, however, that the choice can be equivalently done for the sequence of decisions. This is known as the Principle of Optimality and implies that:

$$v_t(x_t, z_t) = \max_{\{y_j\}_{j=t}^T, y_j\in\Gamma(x_j,z_j)} \left\{\sum_{j=t}^{T} \beta^{j-t+1} u(x_j, y_j, z_j)\right\}. \tag{11}$$

When $t = 1$, the above equation is just the expression in (9). Notice that (11) only holds because we are considering that z_t is a constant in this paragraph, that is, the above corresponds to the nondeterministic case.

We turn now to the stochastic case and extend this recursive definition to this setting by considering that the value function for next period is evaluated through the quantile operator $\mathrm{Q}_\tau[\cdot]$. As before, define $v_{T+1}(x, z) \equiv 0$ and obtain

$$\begin{aligned} v_T(x_T, z_T) &= \max_{y_T\in\Gamma(x_T,z_T)} \{u(x_T, y_T, z_T) + \beta \mathrm{Q}_\tau[v_{T+1}(x_{T+1}, z_{T+1})|z_T]\} \\ &= \max_{y_T\in\Gamma(x_T,z_T)} \mathrm{Q}_\tau\Big[u(x_T, y_T, z_T) + \beta v_{T+1}(x_{T+1}, z_{T+1})\Big|z_T\Big] \\ &= \max_{y_T\in\Gamma(x_T,z_T)} \mathrm{Q}_\tau\Big[u(x_T, y_T, z_T)\Big|z_T\Big] \\ &= \max_{y_T\in\Gamma(x_T,z_T)} u(x_T, y_T, z_T) \\ &= u(x_T, y_T^*, z_T), \end{aligned}$$

where $x_{T+1} = \sigma(x_T, y_T, z_{T+1})$ and $y_T^* \in \Gamma(x_T, z_T)$ is the optimal $y \in \mathcal{Y}$ in the above problem. Notice that the quantile may be dropped in the fourth line above because it is conditioned with respect to z_T and there is no other random variable once z_T is known. When we condition on z_{T-1}, however, z_T is stochastic and the quantile cannot be dropped. We can, however, put terms depending on z_{T-1}, such as $u(x_{T-1}, y_{T-1}, z_{T-1})$ inside a quantile conditioning on z_{T-1}. This is done in the derivation of v_{T-1} below:

$$v_{T-1}(x_{T-1}, z_{T-1}) = \max_{y_{T-1}\in\Gamma(x_{T-1},z_{T-1})} \{u(x_{T-1}, y_{T-1}, z_{T-1}) + \beta \mathrm{Q}_\tau[v_T(x_T, z_T)|z_{T-1}]\}$$

$$= \max_{y_{T-1}\in\Gamma(x_t,z_t)} Q_\tau\left[u(x_{T-1}, y_{T-1}, z_{T-1}) + \beta v_T(x_T, z_T)\Big|z_{T-1}\right]$$
$$= \max_{y_{T-1}\in\Gamma(x_t,z_t)} Q_\tau\left[u(x_{T-1}, y_{T-1}, z_{T-1}) + \beta u(x_T, y_T^*, z_T)\Big|z_{T-1}\right]$$
$$= Q_\tau\left[u(x_{T-1}, y_{T-1}^*, z_{T-1}) + \beta u(x_T, y_T^*, z_T)\Big|z_{T-1}\right]$$
$$= Q_\tau\left[\sum_{j=T-1}^{T} \beta^{j-T+1} u(x_j, y_j^*, z_j)\Big|z_{T-1}\right],$$

where $y_{T-1}^* \in \Gamma(x_{T-1}, z_{T-1})$ is the optimal choice and the next period state follows the law of motion σ. In the above derivation, we are choosing y_{T-1}^* for the period $T-1$, but the Principle of Optimality, mentioned above, shows that this is equivalent to choose jointly (y_{T-1}^*, y_{T-1}^*) in the expression in the last line.

We iterate another period to illustrate the composition of the quantile operator. Thus, adopting the same notation as above, we obtain v_{T-2} as follows:

$$v_{T-2}(x_{T-2}, z_{T-2}) \equiv \max_{y_{T-2}\in\Gamma(x_{T-2},z_{T-2})} \{u(x_{T-2}, y_{T-2}, z_{T-2}) + \beta Q_\tau[v_{T-1}(x_{T-1}, z_{T-1})|z_{T-2}]\}$$
$$= Q_\tau\left[u(x_{T-2}, y_{T-2}^*, z_{T-2}) + \beta v_{T-1}(x_{T-1}, z_{T-1})\Big|z_{T-2}\right]$$
$$= Q_\tau\left[u(x_{T-2}, y_{T-2}^*, z_{T-2}) + \beta\left\{Q_\tau\left[\sum_{j=T-1}^{T} \beta^{j-T+1} u(x_j, y_j^*, z_j)\Big|z_{T-1}\right]\right\}\Big|z_{T-2}\right]$$
$$= Q_\tau\left[Q_\tau\left[\sum_{j=T-2}^{T} \beta^{j-T+2} u(x_j, y_j^*, z_j)\Big|z_{T-1}\right]\Big|z_{T-2}\right].$$

Proceeding in this way, we can obtain v_t, for $t = 1, ., T$ by:

$$v_t(x_t, z_t) = Q_\tau\left[\ \cdots\ Q_\tau\left[\sum_{j=t}^{T} \beta^{j-t} u(x_j, y_j^*, z_j)\Big|z_{T-1}\right]\cdots\ \Big|z_t\right],$$

where the $\cdots$ above represents the application of $T-t$ operators Q_τ, as well as corresponding conditionals on z_j, for $j = t, \ldots, T-1$. It is useful to simplify the above expression using the notation Q_τ^{T-t} to denote the recursively application of these conditional quantiles. That is, the above becomes:

$$v_t(x_t, z_t) = Q_\tau^{T-t}\left[\sum_{j=t}^{T} \beta^{j-t} u(x_j, y_j^*, z_j)\right]. \tag{12}$$

As already mentioned above, the Principle of Optimality, established for recursive quantile preferences by de Castro and Galvao (2019, Proposition 3.17, p.1915) guarantees that the choice of y_j^* for each period $j = t, \ldots, T$ is equivalent to the choice of the whole sequence $\{y_j\}_{j=t}^{T}$ at once.

To sum up, once we set $v_{T+1} \equiv 0$, the sequence of value functions v_t can be recursively defined for $t = T-1, \ldots, 1$, by the following equation:

$$v_t(x_t, z_t) \equiv \max_{y_t\in\Gamma(x_t,z_t)} \left\{u(x_t, y_t, z_t) + \beta Q_\tau\left[v_{t+1}(x_{t+1}, z_{t+1})|z_t\right]\right\}$$
$$= \max_{y_t\in\Gamma(x_t,z_t)} Q_\tau\left[u(x_t, y_t, z_t) + \beta v_{t+1}(x_{t+1}, z_{t+1})\Big|z_t\right], \tag{13}$$

where $x_{t+1} = \sigma(x_t, y_t, z_{t+1})$.

The above sequence of functions are well-defined, as the following result establishes.

Proposition 2. *Under Assumption 1, the recursive definitions (13) are well-defined for $t = T-1, \ldots, 1$.*

Proof. See Appendix A.2. □

The procedure described above leads naturally to an value function iteration algorithm to find the sequence of value functions.

4.2. Algorithm to compute the sequence of value and policy functions

Now we describe how to proceed to find a numerical estimate of the sequence of value functions $v_T, v_{T-1}, ., v_1, v_0$ described above when shocks $\mathcal{Z}$ are discrete. This method is known to work from the Bellman equation to compute the value function by backward iterations on an initial guess. As in the previous section, an important feature of the algorithm is the computation of the conditional τ-quantile, step 3 below, instead of the conditional average.

Since the method is similar to the infinite horizon case, below we only spell out the differences with respect to the algorithm detailed in Section 4.1.

1. **Functional forms and parameters.** This steps follows exactly the one in the infinite case.
2. **Discretize and initialize variables.** It is identical to the infinite case, up to item (e). The other ones should be modified as follows:
   (f) Initialize as constant and equal to zero the sequence of value functions $v_t : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$, for $t = 1, \ldots, T+1$ by a $(T+1) \times p \times k$ matrix $V = (V[t, i_x, i_z])_{(T+1)\times p\times k}$, defined for each $(t, i_x, i_z) \in \{1, \ldots, T+1\} \times \{1, \ldots, p\} \times \{1, \ldots, k\}$,
$$V[t, i_x, i_z] \equiv 0.$$
   (g) Initialize the array $G = \left(G[t, i_x, i_y, i_z, i_w]\right)_{(T+1)\times p\times q\times k\times k}$, which is defined for each element $(t, i_x, i_y, i_z, i_w) \in \{1, \ldots, T+1\} \times \{1, \ldots, p\} \times \{1, \ldots, q\} \times \{1, \ldots, k\} \times \{1, \ldots, k\}$, as $G[t, i_x, i_y, i_z, i_w] = 0$.
   (h) Initialize the array $H = \left(H[i_x, i_y, i_z]\right)_{p\times q\times k}$, defined for each $(i_x, i_y, i_z) \in \{1, \ldots, p\} \times \{1, \ldots, q\} \times \{1, \ldots, k\}$, as $H[i_x, i_y, i_z] = 0$. (This records the quantile of g_t.)
   (i) Initialize the array $Y = (Y[t, i_x, i_z])_{T\times p\times k}$, defined for each $(t, i_x, i_z) \in \{1, \ldots, T\} \times \{1, \ldots, p\} \times \{1, \ldots, k\}$ by $Y[t, i_x, i_z] = 0$. (This records the optimal policy y_t^* at time t as a function of (x, z) defined by (i_x, i_z).)
   (j) Compute the budget set $\Gamma(x, z)$ and record them as $B = (B[i_x, i_y, i_z])_{p\times q\times k}$ defined by:
$$B[i_x, i_y, i_z] = \begin{cases} 1, & \text{if } b^{i_y} \in \Gamma(a^{i_x}, s^{i_z}) \\ 0, & \text{otherwise} \end{cases}$$
   (k) Define $t = T$.
3. **Compute the conditional τ-quantile.** This is almost the same of the infinite case, with the only difference the need to index by t, that is, we should use $G[t, i_x, i_y, i_z, i_w]$ instead of $G[i_x, i_y, i_z, i_w]$. The details are thus omitted.[8]
4. **Compute the maximum value function.** This repeats the infinite case.
5. **Obtain the function v_t.** Finally, calculate function v_t and repeat if $t > 1$ and stop, otherwise
   (a) For each (i_x, i_z), define $V[t, i_x, i_z] = H[i_x, i_y^*, i_z]$.
   (b) If $t > 1$, put $t \leftarrow t - 1$ and repeat from item (3) above; otherwise, stop.

5. Performance of the computational algorithm

In this section, we first assess the accuracy and performance of the numerical algorithm. To do so, we use a quantile recursive model for intertempotal consumption. In this case it is possible to compute an algebraic closed form solution for both the value and policy functions, as a function of given parameters. We, then, for the same set of parameters, simulate the model using the numerical methods discussed previously. Finally, we are able to evaluate the accuracy and performance of the proposed numerical methods by comparing the algebraic and numerical solutions.

We also use the intertemporal consumption model as an example to illustrate the numerical methods by computing the model for different parameters and comparing results with the standard expected utility. Finally, we simulate the model and compute the density functions of the allocation of asset savings for the next period.

5.1. Intertemporal consumption

We now return to the intertemporal consumption-based dynamic model introduced in Section 2.2. This model appears in de Castro et al. (2022b). This example is useful because in this case it is possible derive explicit formulas for the value function, the optimal consumption and asset hold, as well as their corresponding paths. Thus, to assess the accuracy of the proposed algorithm we compare the numerical results with the theoretical closed form solutions.

Recall, from Section 2, the following economy. At the beginning of period t, the decision-maker (DM) has $x_t \in \mathcal{X} \subset \mathbb{R}_+$ units of the risky asset, with return $z_t \in \mathcal{Z} \subseteq \mathbb{R}_{++}$. With wealth $x_t z_t$ at the beginning of period t, the DM decides the number of units y_t of the risk asset, which is equal to the next period's state $y_t = x_{t+1}$ and $c_t = x_t z_t - y_t$ is the amount consumed in period t. From this, the next period how many units of the risky asset x_{t+1} is given by the law of motion $\sigma : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$, as follows:
$$x_{t+1} = \sigma(x_t, y_t, z_{t+1}) = y_t.$$

The dynamic problem of interest is to choose a sequence $y_t = x_{t+1}$ to maximize the following recursive equation:
$$V(x_t, z_t) = \max_{x_{t+1} \in \Gamma(x_t, z_t)} \left\{ U(c_t) + \beta \mathrm{Q}_\tau[V(x_{t+1}, z_{t+1}) | z_t] \right\}, \tag{14}$$

[8] The interpolation discussed in Appendix B can be used for the finite case here as well.

where

$$c_t = x_t z_t - x_{t+1}, \tag{15}$$

$\Gamma(x_t, z_t) = [0, x_t z_t]$ is the budget set, and $U : \mathbb{R}_+ \to \mathbb{R}$ defines the utility function $u(x_t, y_t, z_t) = U(x_t z_t - y_t) = U(c_t)$. Consider the following assumption.

Assumption 2. The following hold:

(i) $\mathcal{X} = [0, \bar{x}] \subseteq \mathbb{R}_+$;
(ii) $\mathcal{Z} \subseteq \mathbb{R}_{++}$;
(iii) $U : \mathbb{R}_{++} \to \mathbb{R}$ is given by

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \tag{16}$$

where $\gamma > 0, \gamma \neq 1$;
(iv) $0 < \beta < \min\{1, \sup_{z\in\mathcal{Z}} z^{\gamma-1}\}$.

Assumption 2 is standard in economic applications. It specifies the utility function and guarantees that the value function converges. The consumption literature has often worked with an isoelastic utility function (constant elasticity of substitution–CES) also known as Constant Relative Risk Aversion (CRRA).

The model in equations (14)–(15), together with Assumption 2, is simple and the DM decides on the stream of consumption, which is equivalent to deciding on the future state $x_{t+1} \in \Gamma(x_t, z_t)$. The optimization problem can be rewritten in the following form

$$V(x_t, z_t) = \max_{x_{t+1}\in[0, x_t z_t]} \left\{ \frac{(x_t z_t - x_{t+1})^{1-\gamma}}{1-\gamma} + \beta Q_\tau [V(x_{t+1}, z_{t+1}) | z_t] \right\}. \tag{17}$$

The economic model in equation (17) is characterized by three parameters: the discount factor (β), the risk attitude (τ), and the parameter in the CES utility function (γ). The interpretation of the parameters is standard. The discount factor characterizes consumer's patience, is used to discount future payments of intertemporal utility functions, and allows to obtain the present value of future consumption. The risk attitude parameter – given by the quantile τ – describes consumer's reluctance to substitute consumption across states of the world under uncertainty and is meaningful even in an atemporal setting. In the dynamic quantile model, the reciprocal of the parameter γ captures the elasticity of intertemporal substitution (EIS). The EIS is defined as elasticity of consumption growth with respect to marginal utility growth.[9] As mentioned previously, an important feature of the recursive quantile model is that it allows for the complete separation of the risk and EIS parameters, while maintaining important properties as dynamic consistency and monotonicity. This is in sharp contrast with the standard expected utility case, where the model is characterized by only two parameters and the risk attitude cannot be disentangled from the EIS.[10]

5.2. Accuracy of numerical solutions

5.2.1. Theoretical closed form solutions

To assess the accuracy of numerical solutions, we first calculate the closed form solutions for the intertemporal consumption model. de Castro et al. (2022b) show that, under Assumptions 1–2, there exists a function $V : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ satisfying (17). Moreover, it is shown that the unique value function is differentiable in x, strictly increasing in z, and the following Euler equation holds:

$$Q_\tau \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} z_{t+1} \middle| z_t \right] = 1. \tag{18}$$

Although one is able to estimate this Euler equation using quantile regression methods, in this paper, we concentrate on computing numerical solutions to the value function directly. Here we also remark that Santos (1999) suggests to bound the approximation error for any arbitrary solution based upon the computation of the Euler equation residuals. In particular, the result asserts that the approximation error of the policy function is of the same order of magnitude as that of the Euler equation residuals, ε, whereas the approximation error of the value function is of order ε^2. In this paper, although we are able to compute the Euler equation as above, we take advantage from the fact of being able to compute explicit analytical solutions for the model and evaluate the accuracy of the numerical methods using these closed form functions.

In fact, de Castro et al. (2022b) derive explicit closed form expressions for the optimal asset allocation and consumption, as well as to the consumption path. In particular, let $r_{\tau,s}(z)$ be defined recursively by

$$r_{\tau,0}(z) = 1,$$

[9] Under time separable utility, this definition is equivalent to the percent change in consumption growth per percent increase in the net interest rate.
[10] See Hall (1978, 1988) for discussions on the separation of these two parameters in the EU case.

$$r_{\tau,s}(z) = r_{\tau,s-1}(\mathrm{Q}_\tau[w|z]) \cdot \mathrm{Q}_\tau[w|z] \quad (s \geqslant 1). \tag{19}$$

Given this, define the function:

$$S(z) \equiv \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} [r_{\tau,s}(z)]^{\frac{1-\gamma}{\gamma}}. \tag{20}$$

Observe that $S(z)$ depends on β, τ and γ.

Proposition 3 (de Castro, Galvao, and Nunes (2022))**.** *Under monotone shocks and Assumption 2, the unique value function $V : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ satisfying (17) is given by*

$$V(x, z) = \frac{1}{1-\gamma} \cdot x^{1-\gamma} \cdot \left[(1 + S(z))^{\gamma} z^{1-\gamma}\right]. \tag{21}$$

Moreover, the optimal asset allocation y^* is interior and given by the policy function $y^* : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y}$:

$$y^* = y^*(x, z) = \frac{zS(z)}{1 + S(z)} \cdot x, \tag{22}$$

so that the consumption is given by

$$c^* = c^*(x, z) = \frac{z}{1 + S(z)} \cdot x. \tag{23}$$

Therefore, the optimal consumption path $\{c_t\}_{t=1}^{\infty}$ is given by

$$c_{t+1} = m_\tau(z_t, z_{t+1}) \cdot c_t, \tag{24}$$

where

$$m_\tau(z_t, z_{t+1}) \equiv \frac{S(z_t)}{1 + S(z_{t+1})} \cdot z_{t+1}. \tag{25}$$

Proposition 3 provides explicit closed form solutions for the value function, and optimal asset allocation and consumption. The optimal policy functions, in equations (22) and (23), are linear functions of x. In particular, the optimal policy rules, asset allocation and consumption, $y^*(x, z)$ and $c^*(x, z)$, are functions of current state x multiplied by a factor that captures the uncertainty, given by the shock z, through the quantile. The uncertainty is resolved through the recursive quantile function $r_{\tau,s}(z)$ in (19). The expressions in (22) and (23) also show that the optimal asset allocation and consumption are functions of the three parameters characterizing the model, the discount factor β, the EIS $1/\gamma$, and the risk attitude (quantile) τ.

If one assumes that the shocks are independent and identically distributed (iid), it is possible to specialize the above results as follows:

Example 2 (The iid case)**.** If the shocks are independent, then $\mathrm{Q}_\tau[w|z]$ becomes a constant, $\mathrm{Q}_\tau[w]$, such that (19) reduces to $r_{\tau,s} = r_{\tau,s}(z) = \mathrm{Q}_\tau[w]^s$. Similarly, let $a_{\tau,\gamma} = \beta^{\frac{1}{\gamma}} (\mathrm{Q}_\tau[w])^{\frac{1-\gamma}{\gamma}}$. Then,

$$1 + \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} [r_{\tau,s}(z)]^{\frac{1-\gamma}{\gamma}} = 1 + \sum_{s=1}^{\infty} a_{\tau,\gamma}^s = 1 + \frac{a_{\tau,\gamma}}{1 - a_{\tau,\gamma}} = \frac{1}{1 - a_{\tau,\gamma}},$$

where $|a_{\tau,\gamma}| < 1$ by Assumption 2. With this, the above results simplify to the following:

$$V(x_t, z_t) = \frac{(1 - a_{\tau,\gamma})^{-\gamma}}{1-\gamma} (x_t z_t)^{1-\gamma}, \tag{26}$$

$$x_{t+1} = a_{\tau,\gamma} x_t z_t, \tag{27}$$

$$c_t = (1 - a_{\tau,\gamma}) x_t z_t. \tag{28}$$

From Example 2 we have closed form solutions for the value function, as well as the asset allocation and consumption. Hence, we are able to compute the value and policy functions numerically and compare them with these theoretical value to assess the performance of the proposed solution algorithm.

5.2.2. Numerical procedures

Now we illustrate the above algorithm using the consumption-based economy example as described above. Codes are written in `Julia 1.8`.

We specify the CRRA utility function, as described in equation (16) above. This is the only known primitive function in the model. Recall that the reciprocal of the parameter γ captures the elasticity of substitution.

For the consumption problem the primitive parameters are: (τ, β, γ). To solve for the value function, we need to assign particular values to these parameters.

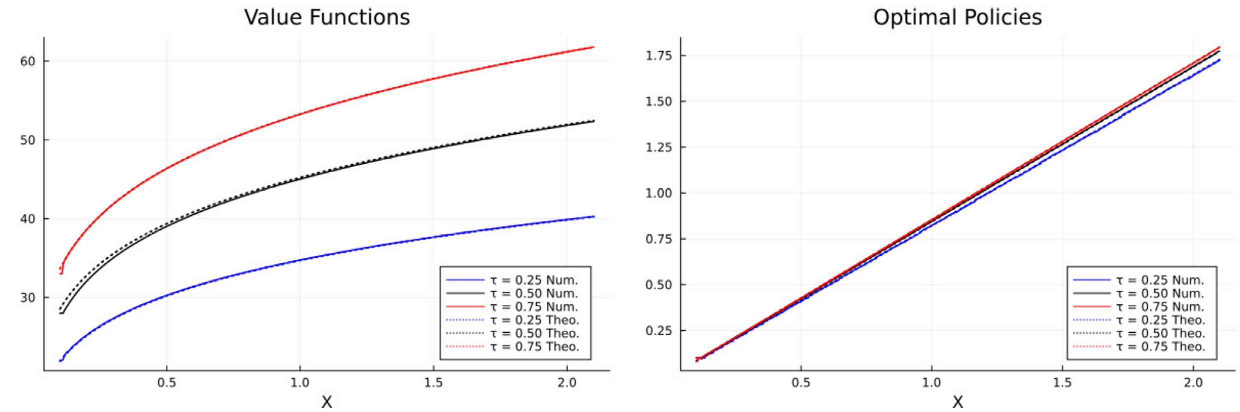

Fig. 1. Value and Policy Function for $\gamma = 0.80$ and Grid $p = q = 250$.

1. We choose the model and the functional form for the utility function $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$, the budget set $\Gamma : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y}$, the transition function $\sigma : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ as following:

$$u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R} \text{ given by } u(x_t, y_t, z_t) = \frac{(x_t z_t - y_t)^{1-\gamma}}{1-\gamma};$$
$$\Gamma : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y} \text{ given by } \Gamma(x_t, z_t) = [0, x_t z_t];$$
$$\sigma : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X} \text{ given by } \sigma(x_t, y_t, z_{t+1}) = y_t.$$

2. Represent the functions through matrices or arrays and initialize the variables of the algorithm:
   (a) Consider a discretization for $\mathcal{X} = \{x^1, \dots, x^p\}$ and $\mathcal{Y} = \{y^1, \dots, y^q\}$ from zero to two with increments $1/p$ and $1/q$. We set different values for $p = q \in \{250, 500, 1000\}$.
   (b) We consider the following the stochastic transition matrix $P[i, j] = p_{ij}$:

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} \\ p_{51} & p_{52} & p_{53} & p_{54} & p_{55} \end{bmatrix} = \begin{bmatrix} 0.25 & 0.15 & 0.15 & 0.25 & 0.20 \\ 0.25 & 0.15 & 0.15 & 0.25 & 0.20 \\ 0.25 & 0.15 & 0.15 & 0.25 & 0.20 \\ 0.25 & 0.15 & 0.15 & 0.25 & 0.20 \\ 0.25 & 0.15 & 0.15 & 0.25 & 0.20 \\ 0.25 & 0.15 & 0.15 & 0.25 & 0.20 \end{bmatrix}. \tag{29}$$

   Shocks assume the following values: $\mathcal{Z} = \{0.90, 0.95, 1, 1.05, 1.15\}$.
3. We specify the parameters as following: the discount factor $\beta = 0.95$; and quantile risk attitudes as $\tau = \{0.25, 0.5, 0.75\}$. We vary the EIS parameter as follows $\gamma = \{0.8, 1.25\}$, such that the EIS $= 1/\gamma = \{1.25, 0.8\}$.

Notice that the stochastic transition matrix, P, in equation (29) describes an iid process, such that we are able to compare the numerical results with the theoretical functions in Example 2. Also, we choose $\mathcal{Z}$ as given above such that Assumption 2-(iv) is satisfied for both EIS parameters.

The accuracy evaluation procedure is as follows. First, using the specifications in the intertemporal consumption example above, for each point in the grid, we calculate the theoretical closed form solutions for value and policy functions in equations (26) and (27), respectively. Second, we apply the algorithm described in Section 3 to compute the same functions numerically. Finally, we calculate the approximation error by subtracting the numerical solutions from the closed form solutions. To evaluate the accuracy of the numerical computations we calculate the average statistic of the approximation error over the evaluation points, as in Judd et al. (2017). We present results in levels, and after standardizing the error, such that the statistic is unit free. In latter case, we divide the difference by the value of the function under closed form solution. Results are given in the next section.

5.2.3. Results

Results for assessing the accuracy of the numerical procedures relative to the theoretical derivations are provided for the two EIS values. Figures 1–3 collect the results for $\gamma = 0.80$, and Figures 4–6 display results for $\gamma = 1.25$, when varying the grid. To avoid divergence we exclude zero from the grid. In each figure, we plot value functions and policy functions for the allocation of asset savings for the next period computed with the numerical simulations in solid lines, and the theoretical closed form solutions in dotted lines. Left panels show results for value functions, and right panels for policy functions. Each panel collects results for the three quantiles, $\tau = \{0.25, 0.50, 0.75\}$, in blue, black, and red colors, respectively. We use five different shock values in each model, but present results only for the first shock. Results for the other cases are qualitatively

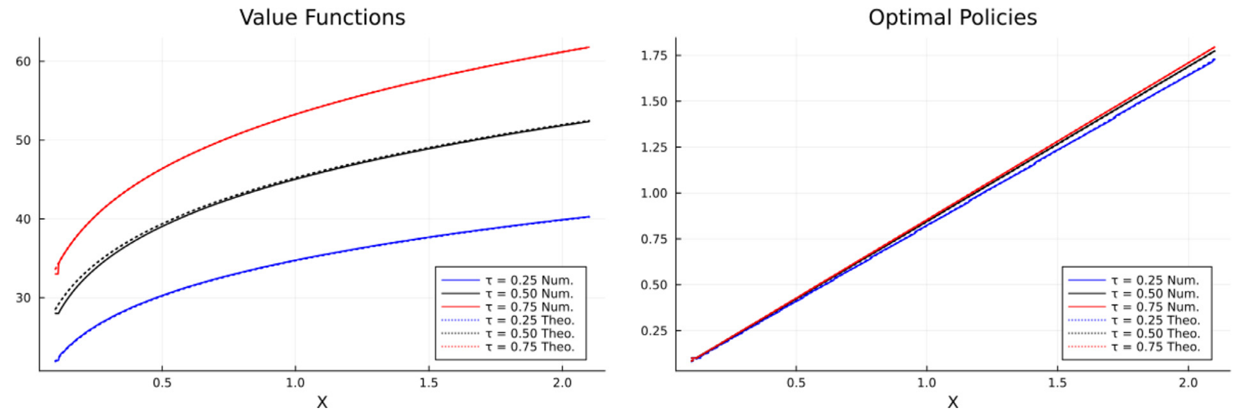

Fig. 2. Value and Policy Function for 0.80 and Grid $p = q = 500$.

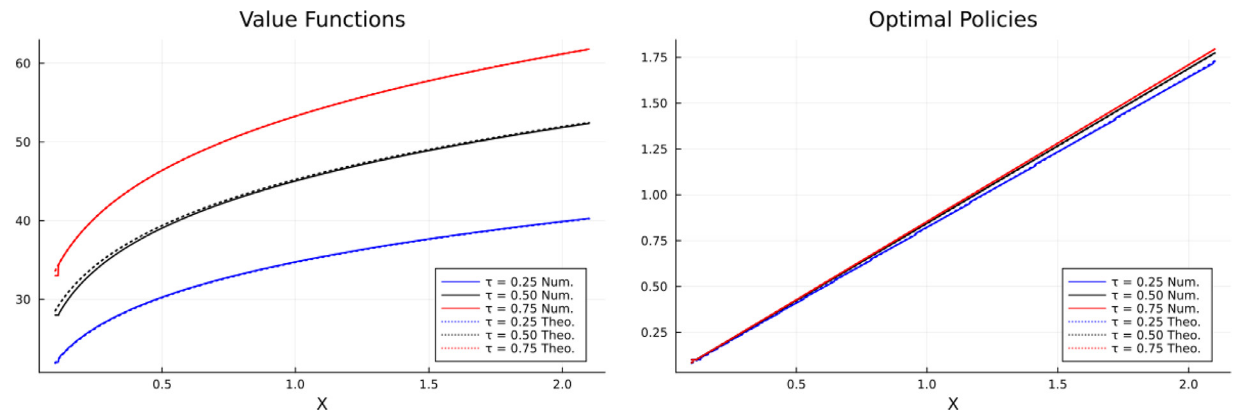

Fig. 3. Value and Policy Function for $\gamma = 0.80$ and Grid $p = q = 1000$.

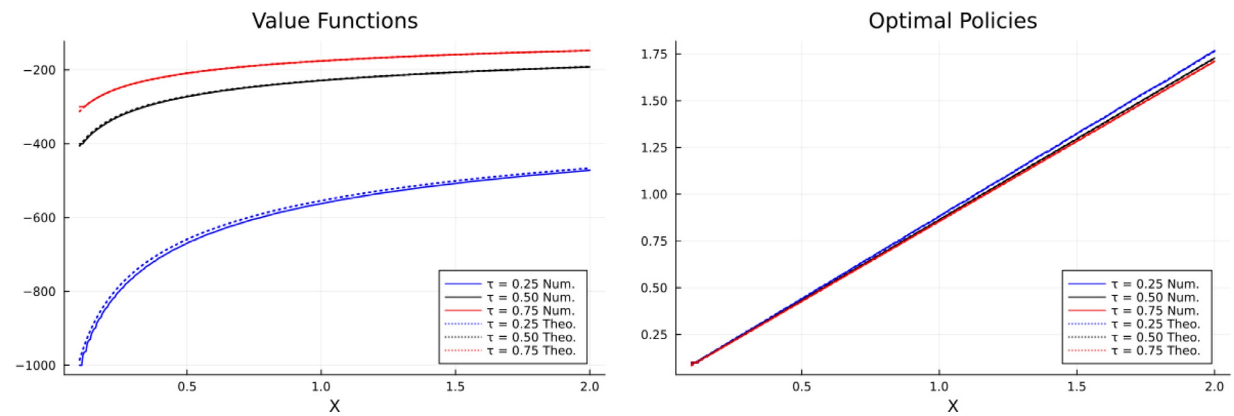

Fig. 4. Value and Policy Function for $\gamma = 1.25$ and Grid $p = q = 250$.

similar and we omit them for brevity. Table 1 below, computes the average statistic of the difference between the theoretical and numerical for value and policy functions.

Figure 1 shows results for the value functions (left panel) and asset allocation policy functions (right panel) when the grid uses only $p = q = 250$. We first note that the numerical value and policy functions (solid lines) are close to their theoretical counterparts (dotted lines). Figure 1 shows that the numerical procedures already approximate their corresponding counterparts very closely even for the smaller grid.

Both panels in Figure 1 display value and policy functions for each of the three quantiles, for both the theoretical and numerical cases. The figure shows that both value and policy functions, for a given X, are increasing over quantiles, that is, the (value and policy) curves for $\tau = 0.25$ are the lowest functions, followed by $\tau = 0.50$, then by $\tau = 0.75$. Thus, for

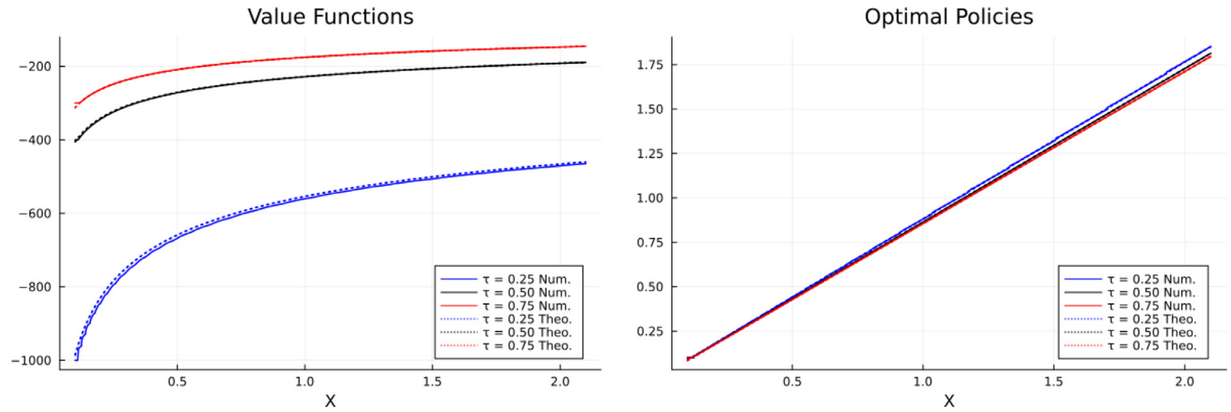

Fig. 5. Value and Policy Function for $\gamma = 1.25$ and Grid $p = q = 500$.

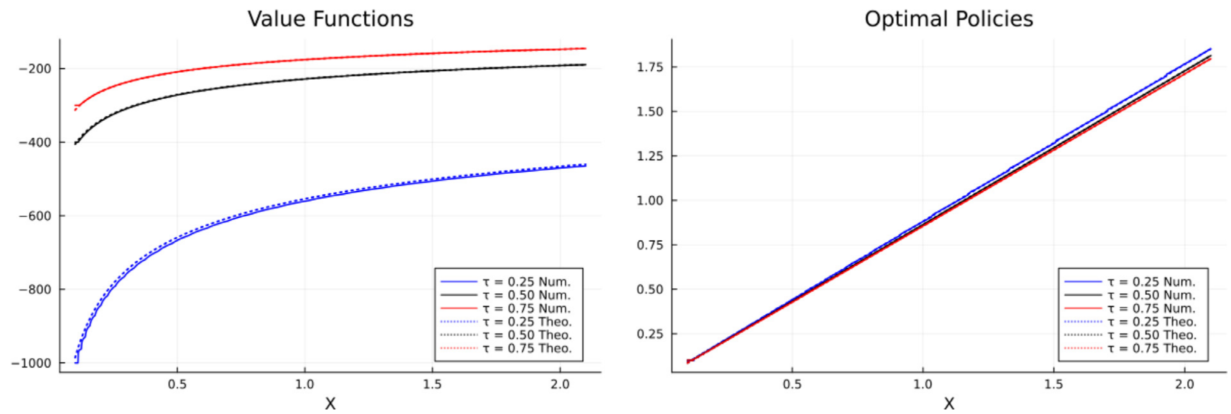

Fig. 6. Value and Policy Function for $\gamma = 1.25$ and Grid $p = q = 1000$.

Table 1
Accuracy of numerical algorithm to compute value and policy functions.

			Value Function		Policy Function	
γ	Grid	τ	L_1-Level	L_1-Normalized	L_1-Level	L_1-Normalized
1.25	250	0.25	8.6968	-0.0146	0.0001	0.0002
		0.50	0.8960	-0.0033	0.0003	-0.0010
		0.75	-0.0451	0.0001	0.0001	-0.0006
	500	0.25	-7.2374	-0.0121	0.0001	0.0001
		0.50	-0.6439	-0.0023	0.0003	-0.0010
		0.75	0.0577	0.0002	0.0001	-0.0007
	1000	0.25	-6.8225	-0.0114	0.0001	0.0000
		0.50	-0.5780	-0.0021	0.0002	-0.0009
		0.75	0.0657	0.0002	0.0001	-0.0007
0.8	250	0.25	-0.0500	-0.0015	-0.0004	-0.0003
		0.50	0.2626	0.0067	0.0012	-0.0028
		0.75	0.0222	0.0005	0.0000	-0.0007
	500	0.25	0.0535	-0.0016	-0.0003	-0.0004
		0.50	-0.2479	0.0063	0.0012	-0.0028
		0.75	-0.0185	0.0004	0.0000	-0.0007
	1000	0.25	-0.0543	-0.0016	-0.0003	-0.0004
		0.50	0.2452	0.0062	0.0012	-0.0027
		0.75	0.0168	0.0004	0.0000	-0.0007

Note: The statistics L_1-Level and L_1-Normalized are, respectively, the average over evaluation points of the approximation error, and the normalized approximation error; γ is the EIS; Grid is the size of the grid used to evaluate the methods; τ is the coefficient of risk attitude.

$\gamma = 0.8$ (EIS $= 1.25$), the larger the risk attitude, the larger the value and policy functions. In addition, the right panel of Figure 1 shows that the optimal asset allocations are linear and monotone increasing over X, as prescribed by the theory.

Figure 2 displays the value and policy functions, in the left and right panels, respectively, for a grid of 500. The figure shows the same patterns as in the previous figure. The left panel of Figure 2 provides evidence that as the grid becomes finer, for all quantiles, the numerical value functions approximation to the theoretical values improve. Moreover, the right panel shows that all three policy functions approximate their corresponding theoretical counterparts closely.

Finally, Figure 3 increases the grid further to p=q=1000. In this case, we observe that the numerical methods approximate the theoretical values very closely. The patterns are similar to those in the previous figures.

Now we investigate the numerical procedures when $\gamma = 1.25$, and hence EIS $= 0.80$. All the other parameters are the same, relative to the previous figures. Figures 4–6 display results for grids 250, 500, and 1000, respectively. The first main finding in Figures 4–6 is that the patterns are similar to those in Figures 1–3 in terms of approximations of the numerical methods relative to their corresponding theoretical counterparts. Numerical approximations are close to theoretical counterparts even for the smallest grid, and improve as the grid becomes finer.

The left panels in Figures 4–6 show that value functions are monotone increasing over X for all three quantiles. Moreover, as in the previous case, the value functions for $\gamma = 1.25$ are also increasing over quantiles, that is $\tau = 0.25$ is the lowest curve, followed by $\tau = 0.50$, and then $\tau = 0.75$.

Now we note that, when γ increases to 1.25, the EIS reduces, in particular, for this case the EIS is smaller than one (EIS $= 1/\gamma = 0.80$). Recall that the EIS measures how willing individuals are to substitute intertemporally between consumption this period and consumption next period. As discussed previously, and displayed in the right panels of Figures 4–6, the policy functions are monotone increasing across X for all quantile cases. However, differently from the case $\gamma = 0.80$, now the order of the policy function has changed over the quantiles. For this case, $EIS < 1$, the more risk averse agent, $\tau = 0.25$, has the highest policy function, for a given value of X, followed by the median, and $\tau = 0.75$ being the lowest policy for a given value of X. Thus, when EIS is smaller than one, and economic agents are less willing to substitute consumption this period and next period – or less sensitivity of consumption growth to changes in interest rates – a more risk averse agent invests more, relative to a less risk averse agent.

The result on the change of order in the policy functions for the allocation of asset savings for the next period across quantiles when the EIS is reduced (from $1/\gamma = 1.25$ to $1/\gamma = 0.80$) is interesting. As EIS decreases, consumption growth is less responsive to changes in interest rates. Everything else equal, if consumption is less responsive to changes, the asset allocation is more responsive. To understand and evaluate this change, we use the asset allocation policy function in equation (27). We wish to assess how $a_{\tau,\gamma}$ varies with γ. Recall that $a_{\tau,\gamma} = \beta^{\frac{1}{\gamma}}(\mathrm{Q}_\tau[w])^{\frac{1-\gamma}{\gamma}}$. Define $l(a_{\tau,\gamma}) = \ln(a_{\tau,\gamma}) = \frac{1}{\gamma}\ln\beta + \frac{1-\gamma}{\gamma}\ln \mathrm{Q}_\tau[w]$. Then,

$$\frac{\partial l(a_{\tau,\gamma})}{\partial\gamma} = -\frac{1}{\gamma^2}[\ln\beta + \ln(\mathrm{Q}_\tau[w])].$$

Observe that for $\beta = 0.95$, $\ln(\beta) = -0.05129$, and $\mathrm{Q}_\tau[w] = \{0.9, 1, 1.05\}$, for $\tau = \{0.25, 0.50, 0.75\}$, respectively, such that $\ln(\mathrm{Q}_\tau[w]) = \{-0.105, 0, 0.049\}$. Thus, $[\ln\beta + \ln(\mathrm{Q}_\tau[w])] < 0$ and the above derivative becomes positive for all three quantiles. Hence, increasing γ implies increasing $a_{\tau,\gamma}$, which in turn implies increasing asset allocation. Moreover, we see that the magnitude of the effect on $\tau = 0.25$ is relatively larger than that on the median, which is lager than that on $\tau = 0.75$. This shows that the effect of increasing the EIS is positive for all quantiles, but it is relatively larger for smaller quantiles, which induces the change in order in the policy functions.

Table 1 presents the results for the accuracy error of the numerical methods. Recall that the average approximation error is calculated by the difference between the numerical and closed form solutions of value and policy functions for each point that these functions are evaluated, and then taking the average of the approximation error over the evaluation points. We present results for both the comparison in levels and after standardizing the calculation of the approximation error. Both versions of comparisons deliver the high levels of accuracy for the value function iteration computation of the quantile model. These results show evidence that the numerical procedures are able to approximate the theoretical solutions very closely. Moreover, the results corroborate the findings displayed in the figures above.

Overall, results from these exercises show evidence that the suggested numerical methods to solve the dynamic quantile model provide a high degree of accuracy. Hence, researchers applying this solution algorithm can be confident that their quantitative answers are sound. Moreover, since the dynamic quantile model allows for separation of the risk attitude, τ, and the EIS, $1/\gamma$, investigating the effects of varying these parameters, together or separately, provides interesting insights when studying dynamic economic models under uncertainty.

5.3. Comparison with the expected utility

To illustrate the methods further, we compute the dynamic quantile model numerically varying several of the parameters, and compare results with the standard expected utility (EU) case.

5.3.1. Numerical procedures

In this section we simulate the model using an extension of the algorithm with interpolation. The procedure is described in the Appendix B. We use the same specification for the variables $\mathcal{X}$, $\mathcal{Y}$, and the CRRA utility function, as in the previous

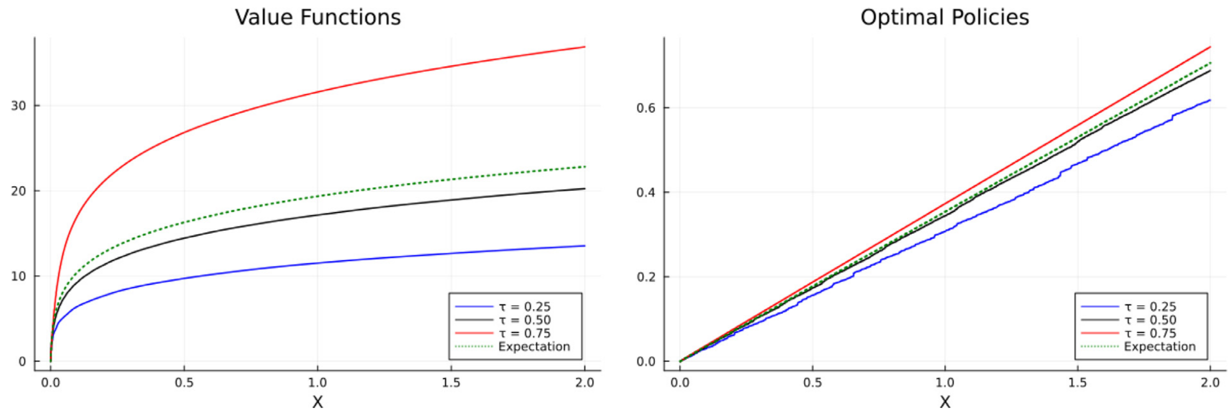

Fig. 7. Value and Policy Functions for $\gamma = 0.80$ and iid shocks.

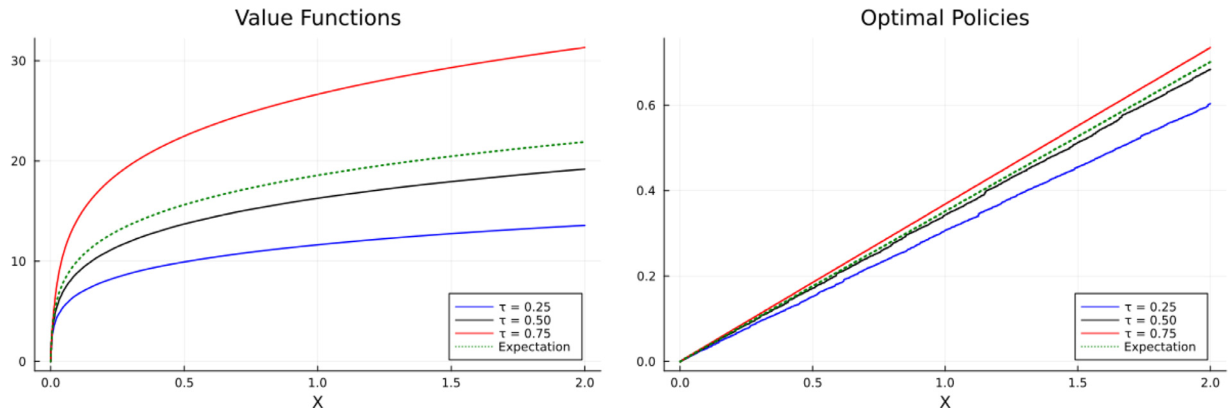

Fig. 8. Value and Policy Functions for $\gamma = 0.80$ and Markov shocks.

simulations. We also fix the discount factor β, EIS, and the risk attitudes τ's parameters as following:

$$\beta = 0.95; \quad \gamma = \{0.80, 1.25\}; \quad \tau \in \{0.25, 0.50, 0.75\},$$

such that $\text{EIS} = 1/\gamma = \{1.25, 0.8\}$. We fix the grid for the variables $\mathcal{X}$, $\mathcal{Y}$ with $p = q = 500$ increments for all computations. We use the same specifications for the EU case, with the exception that it only relies on β and γ.

To make the results easier to visualize, we set the values for the shocks as: i) $\mathcal{Z} = \{0.4, 0.6, 0.8, 1, 1.2\}$ for $\gamma = 0.80$; ii) $\mathcal{Z} = \{0.9, 1, 1.2, 1.4, 1.6\}$ for $\gamma = 1.25$. We present results for the iid and Markov cases varying the transition matrices. For the iid case we use the transition matrix as:

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} \\ p_{51} & p_{52} & p_{53} & p_{54} & p_{55} \end{bmatrix} = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix}. \tag{30}$$

For the Markov case we consider the following the stochastic transition matrix $P[i, j] = p_{ij}$:

$$P = \begin{bmatrix} 0.1 & 0.2 & 0.4 & 0.2 & 0.2 \\ 0.2 & 0.1 & 0.4 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.1 & 0.3 & 0.1 \\ 0.3 & 0.2 & 0.2 & 0.1 & 0.2 \\ 0.2 & 0.25 & 0.25 & 0.2 & 0.1 \end{bmatrix}. \tag{31}$$

5.3.2. Results

Now we present the results. Figures 7 and 8 collect results for $\gamma = 0.80$, for the iid and Markov shocks, respectively. Quantiles $\tau = 0.25$, $\tau = 0.50$, and $\tau = 0.75$ are solid lines in blue, black and red, respectively. The standard expected utility is the green dotted line.

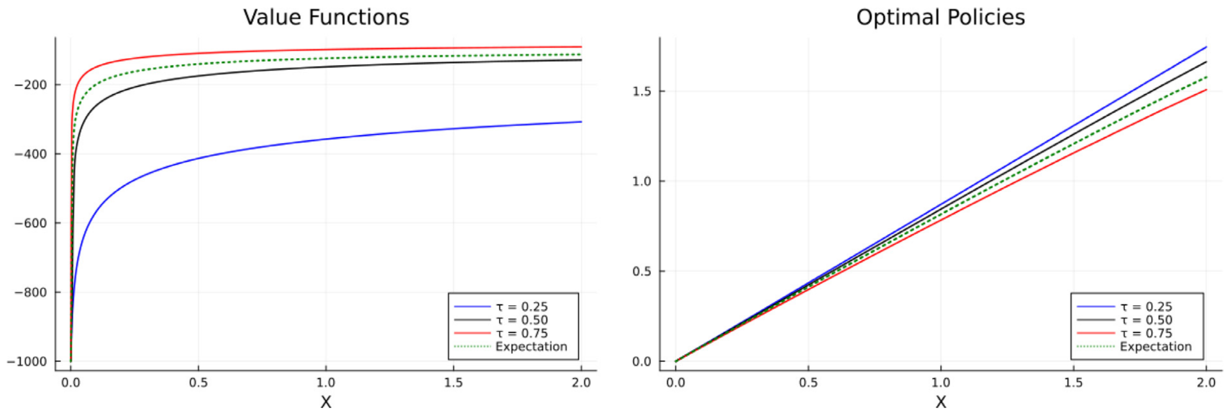

Fig. 9. Value and Policy Functions for $\gamma = 1.25$ and iid shocks.

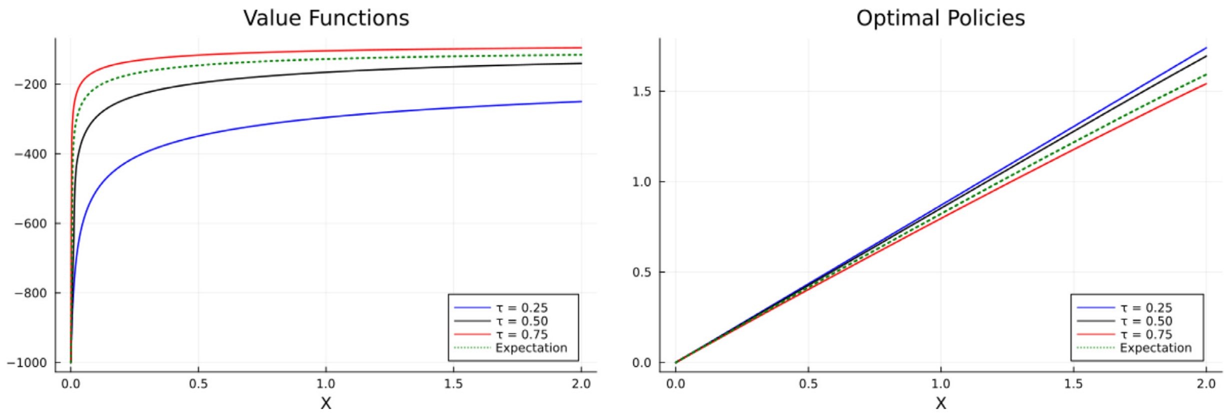

Fig. 10. Value and Policy Functions for $\gamma = 1.25$ and Markov shocks.

Figures 7 and 8 display results for the value functions in the left panels, and the asset allocation policy function in the right panels. The figures show that results for the iid case are qualitative similar to those for the Markov case. Both value and policy functions are monotone increasing across X, for quantile and expectation models. In addition, the right panels in Figures 7 and 8 show that, for a given X, the optimal asset allocation level is increasing on the risk attitude, τ, i.e., the more risk taker – larger quantiles –, the larger the asset allocation when $\gamma = 0.80$ (EIS $= 1/\gamma = 1.25$).

We also notice that, the results for the EU are close to the median case. The right panels in Figures 7 and 8 show the asset allocation policy functions for the expectation are only slightly below the median, for both the iid and Markov cases.

Now we move to the case $\gamma = 1.25$, and hence EIS $= 1/\gamma = 0.8$. Figures 9 and 10 report results for the iid and Markov shocks, respectively. We first observe the same monotone increasing across X patters for both the value and policy functions for all three quantiles and the expected utility. Moreover, policy functions for the allocation of asset savings for the next period are linear for both quantile and EU.

From the right panels in Figures 9 and 10, we find a larger asset allocation level for higher risk aversions, that is, for a given X, the asset allocation curve for $\tau = 0.25$ is above $\tau = 0.50$ and $\tau = 0.75$, respectively. Thus, as EIS increases, and willingness to intertemporally substitute consumption this period and consumption next period increases, the more risk averse agent has a larger asset allocation relative to low risk averse.

Overall, these simulations show evidence that the numerical procedures are able to handle iid and Markov cases well. Moreover, these exercises illustrate an advantage of the dynamic quantile model with respect to the standard EU model, such that, for a given EIS, the figures collect behavior for three different risk attitudes.

5.4. Simulations

Applied economists often characterize the behavior of economic models through statistics from simulated paths of the economy. In this section, we simulate the quantile and EU models, for 10,000 periods. In both cases we use the decision rules for EIS as $\gamma \in \{0.80, 1.25\}$, and for the quantile case each of the three quantiles capture risk aversion $\tau \in \{0.25, 0.50, 0.75\}$. We discard the first 1000 periods as a burn-in to eliminate the transition from the initial starting point. The remaining observations constitute a sample from the ergodic distribution of the economy. We use the same parametrization as that

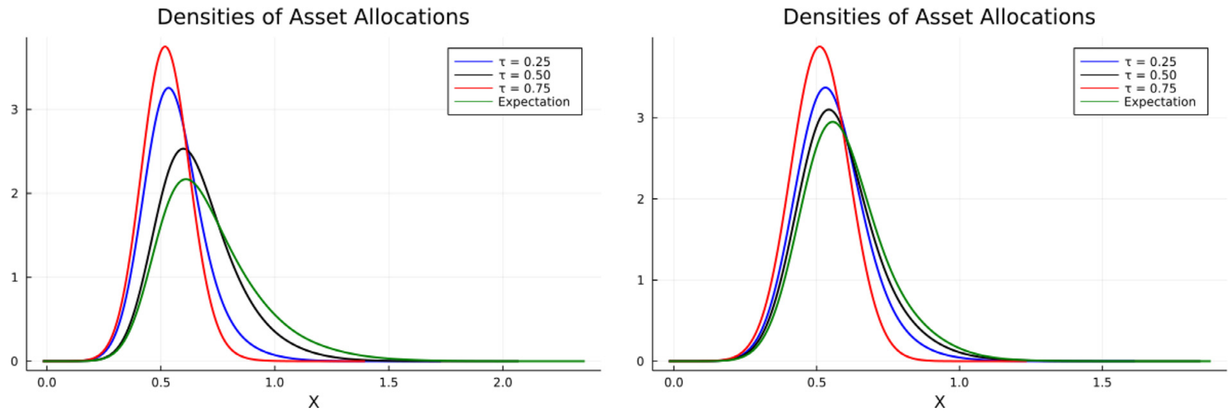

Fig. 11. Densities for the allocation of asset savings for the next period for $\gamma = 0.80$; iid (left panel) and Markov (right panel) shocks.

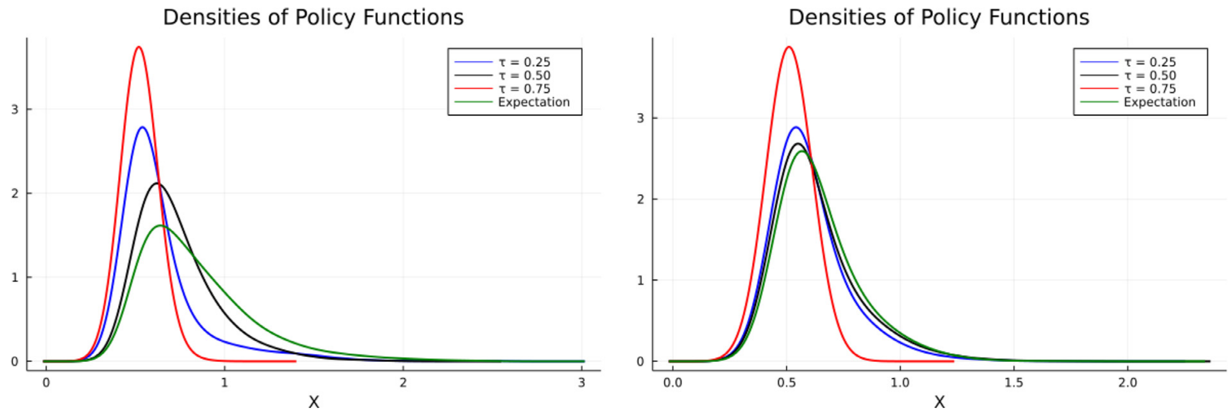

Fig. 12. Densities for the allocation of asset savings for the next period for $\gamma = 1.25$; iid (left panel) and Markov (right panel) shocks.

Table 2
Mean and standard error of the allocation of asset savings for the next period.

	$\gamma = 0.80$ iid		$\gamma = 0.80$ Markov		$\gamma = 1.25$ iid		$\gamma = 1.25$ Markov	
	Mean	SE	Mean	SE	Mean	SE	Mean	SE
$\tau = 0.25$	0.562	0.092	0.552	0.080	0.640	0.240	0.605	0.148
$\tau = 0.50$	0.659	0.147	0.5799	0.105	0.730	0.217	0.629	0.166
$\tau = 0.75$	0.524	0.042	0.5129	0.026	0.524	0.043	0.513	0.026
Expectation	0.709	0.194	0.5989	0.114	0.838	0.310	0.644	0.162

in Section 5.2.2 above. In particular, $\mathcal{Z} = \{0.90, 0.95, 1, 1.05, 1.15\}$, for iid shocks the transition matrix is in (29), and for Markov shocks the matrix is in (31). We initialize simulations from the middle state space. In addition, since the shock in the model is multiplicative, we consider a discretization for $\mathcal{X}$ and $\mathcal{Y}$ from 0.5 to 2. We present results for grid of p=q=500, but results for grid of 1000 are similar.

Figures 11 and 12 report results for the densities of the allocation of asset savings for the next period for $\gamma = 0.80$ and $\gamma = 1.25$, respectively. The left panels display results for iid shocks and the right panels for Markov shocks. Results for quantiles $\tau = 0.25$, $\tau = 0.50$, and $\tau = 0.75$ are in solid blue, black, and red lines, respectively. Results for the EU case are in solid green lines. Moreover, Table 2 collects simple descriptive statistics, mean and standard errors, on these simulated asset allocations.

The left panel in Figure 11 shows that for EIS larger than one, $\gamma = 0.8$, the EU and median cases have larger means and variances. The right panel shows that in the Markov case, all densities are roughly centered around the same value.

The left panel in Figure 12 shows that when EIS decreases, the mean of the distribution increases, relative to the previous case. We can also see that the dispersion $\tau = 0.25$ and $\tau = 0.50$ slightly increase, but for $\tau = 0.75$ and expected utility case the dispersion decreases. The right panel of Figure 12 shows a similar pattern as in the previous case with the more concentrated, although when EIS decreases the average asset allocation increases.

6. Conclusion

This paper studies dynamic programming for quantile preference models, in which the agent maximizes the stream of the future τ-quantile utilities, for $\tau \in (0,1)$. We, first, extend existing theoretical results to allow the dynamic quantile model to have a finite-horizon. Second, we discuss numerical methods, based on value function iterations, for solving the quantile recursive model dynamic programming and compute value and policy functions. Hence, this paper contributes to the literature by offering numerical algorithms that can be used by applied economists to compute and simulate models based on quantile maximization.

To illustrate the developments, we use an intertemporal consumption quantile model, which allows one to calculate explicit algebraic closed form solutions for the value and policy functions. We assess the accuracy of the proposed numerical methods by computing and comparing the theoretical and numerical value and policy functions, for several combinations of the parameters – discount factor, elasticity of intertemporal substitution, and risk attitude, which is measured by the quantiles. The results document evidence that the suggested algorithm provides solutions that are very close to the theoretical counterparts and provide a reliable computational method to solve the problem.

Many issues remain to be investigated. From the theoretical viewpoint, extending the finite horizon model to an overlapping generations framework is an interesting avenue. From the numerical point of view, extensions of the methods to a multigrid scheme as in Chow and Tsitsiklis (1991) would be of interest. Moreover, additional techniques to solve the problem numerically and improve speed and accuracy, as for example, perturbation and Chebyshev polynomials methods, would be welcome advances.

Appendix A. Proofs

A1. Proofs of section 3

We organize the proof of Lemma 1 in a series of Lemmas. Assume Assumption 1 for Lemmas 2–4.

Lemma 2. *If $v \in \mathcal{C}$, the map $(y,z) \mapsto \mathrm{Q}_\tau[v(\sigma(x,y,w),w)|z]$ is continuous.*

Proof. This result is proved in de Castro et al. (2022b). □

Lemma 3. *For each $v \in \mathcal{C}$ the supremum in equation (5) is attained and $\mathbb{T}_\tau(v) \in \mathcal{C}$. Moreover, for each $v \in \mathcal{C}$, the optimal correspondence $\Upsilon_v : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{X}$ defined by*

$$\Upsilon_v(x,z) \equiv \arg\max_{y\in\Gamma(x,z)} \mathrm{Q}_\tau[u(x,y,z) + \beta v(\sigma(x,y,w),w)|z] \tag{A.1}$$

is nonempty and upper semi-continuous.

Proof. Let $g : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ be defined by

$$g(x,y,z,w) \equiv u(x,y,z) + \beta v(\sigma(x,y,w),w), \tag{A.2}$$

where $w \in \mathcal{Z}$ stands for next period's shock and $z \in \mathcal{Z}$ for the current period shock. By Lemma A.2 in de Castro and Galvao (2019, p. 1927), we have that $\mathrm{Q}_\tau[g(x,y,z,w)|z] = u(x,y,z) + \beta \mathrm{Q}_\tau[v(\sigma(x,y,w),w)|z]$. By Lemma 2 above, $\mathrm{Q}_\tau[g(x,y,z,w)|z]$ is continuous in (x,y,z). By Assumption 1, $\Gamma(x,z)$ is compact. From the Berge's Maximum Theorem, the maximum is attained, the value function $\mathbb{T}_\tau(v)$ is continuous and Υ_v is nonempty and upper semi-continuous. Therefore, $\mathbb{T}_\tau(v) \in \mathcal{C}$. □

We conclude the proof of Lemma 1 with the following lemma, which shows that $\mathbb{T}_\tau$ satisfies Blackwell's sufficient conditions for a contraction.

Lemma 4. *$\mathbb{T}_\tau$ satisfies the following conditions:*

(a) *For any $v, v' \in \mathcal{C}$, $v \leqslant v'$ implies $\mathbb{T}_\tau(v) \leqslant \mathbb{T}_\tau(v')$.*
(b) *For any $a \geqslant 0$ and $x \in X$, $\mathbb{T}_\tau(v+a)(x) \leqslant \mathbb{T}_\tau(v)(x) + \beta a$, with $\beta \in (0,1)$.*

Then, $\|\mathbb{T}_\tau(v) - \mathbb{T}_\tau(v')\| \leqslant \beta\|v - v'\|$, that is, $\mathbb{T}_\tau$ is a contraction with modulus β. Therefore, $\mathbb{T}_\tau$ has a unique fixed-point $v_\tau \in \mathcal{C}$.

Proof. To see (a), let $v, v' \in \mathcal{C}$, $v \leqslant v'$ and define g as in equation (A.2) and analogously for g', that is, $g'(x,y,z,w) = u(x,y,z) + \beta v'(\sigma(x,y,w),w)$. It is clear that $g \leqslant g'$. Then, by Lemma A.1 part (vi) of de Castro and Galvao (2019, p. 1926), $\mathrm{Q}_\tau[g(\cdot)|z] \leqslant \mathrm{Q}_\tau[g'(\cdot)|z]$, which implies (a).

To verify (b), we use the monotonicity property (Lemma A.2 of de Castro and Galvao (2019, p. 1927)):

$$\mathrm{Q}_\tau[u(x,y,z) + \beta(v(\sigma(x,y,w),w) + a)|z] = \mathrm{Q}_\tau[u(x,y,z) + \beta v(\sigma(x,y,w),w)|z] + \beta a.$$

Thus, $\mathbb{T}_\tau(v+a) = \mathbb{T}_\tau(v) + \beta a$, that is, (b) is satisfied with equality. The conclusion follows by standard arguments. See, for instance, de Castro and Galvao (2019). □

Proof of Proposition 1: Under Assumption 1, from equation (6) and $\mathbb{T}_\tau(v_\tau) = v_\tau$,

$$\|v^n - v_\tau\| = \|\mathbb{T}_\tau(v^{n-1}) - \mathbb{T}_\tau(v_\tau)\| \leqslant \beta\|v^{n-1} - v_\tau\|.$$

Repeating the same argument n times, we obtain $\|v^n - v_\tau\| \leqslant \beta^n\|v^0 - v_\tau\|$. □

A2. Proofs of section 4

Proof of Proposition 2: Since $\Gamma(x_t, z_t)$ is compact and this correspondence is continuous, with u, σ and v_{t+1} continuous, the optimal $y_t \in \Gamma(x_t, z_t)$ exists and the value function v_t is continuous. □

Appendix B. Interpolation Algorithm to Compute the Conditional Quantile

When solving the value function or the policy function numerically, one often has to calculate the value of these functions outside of the points of the pre-specified grid. In this appendix, we show how to modify the algorithm given in Sections 3.2 and 4.2 to obtain the conditional quantile as an interpolation. For that, it is enough to modify item (3)-(d) and forward in the original algorithm, as follows:

(d) For each (i_x, i_y, i_z), find the only $j \in \{1, ., k\}$ that satisfies:[11]

$$P[i_z, i_w^{(1)}] + \cdots + P[i_z, i_w^{(j-1)}] < \tau \leqslant P[i_z, i_w^{(1)}] + \cdots + P[i_z, i_w^{(j)}].$$

(e) Using the j obtained in the previous step, define $P_l = P[i_z, i_w^{(1)}] + \cdots + P[i_z, i_w^{(j-1)}]$ and $P_u = P[i_z, i_w^{(1)}] + \cdots + P[i_z, i_w^{(j)}]$.[12]

(f) Find α such that $\alpha P_l + (1-\alpha)P_u = \tau$, that is,

$$\alpha = \frac{P_u - \tau}{P_u - P_l}.$$

(g) For each $(i_x, i_y, i_z) \in \{1, \ldots, p\} \times \{1, \ldots, q\} \times \{1, \ldots, k\}$, use the above α and j to define the following interpolation as a proxy of the τ-quantile of G:

$$H[i_x, i_y, i_z] = \alpha G[i_x, i_y, i_z, i_w^{(j-1)}] + (1-\alpha) G[i_x, i_y, i_z, i_w^{(j)}].$$

The rest of the algorithm can continue as before.

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[11] Recall that $\tau \in (0, 1)$ and $P[i_z, i_w^{(1)}] + \cdots + P[i_z, i_w^{(k)}] = 1$. Let $P[i_z, i_w^{(1)}] + \cdots + P[i_z, i_w^{(j-1)}] = 0$ if $j = 1$.

[12] The indexes l and u are mnemonics for *lower* and *upper*.

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