Affiliation, Equilibrium Existence and The Revenue Ranking of Auctions*

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Abstract

Affiliation was introduced by Milgrom and Weber (1982) through the positive dependence intuition, that is, high value of one bidder's estimate makes high values of the others' estimates more likely. However, positive dependence has many alternative definitions; we show that affiliation is one of the most restrictive among them. This poses the question whether affiliation's main implications (equilibrium existence and the revenue ranking of auctions) remain valid for other formalizations of positive dependence. We show that both implications can indeed be generalized in the context of private values, and give counterexamples for further generalizations.

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1 Introduction

Asymmetric information is a central theme in modern economics, not only in game theory, but also in industrial organization, general equilibrium, group decision, finance and many other subdisciplines. Most models assume that each agent privately knows a random variable, and these random variables are statistically independent. Although independence is convenient for theoretical manipulations, it is considered a restrictive and unrealistic assumption. Independence is regarded as restrictive because it is satisfied by a "knife-edge" set of distributions, and unrealistic because there are many potential sources of correlation in the real world: media, education, culture or even evolution. Perceiving these limitations early on, economists tried to surpass the mathematical difficulties and include statistical dependence in their models.

The introduction of affiliation was a milestone in the study of dependence in economics. This remarkable contribution was made by Milgrom and Weber (1982a), who borrowed a statistical concept (multivariate total positivity of order 2, MTP_2) and applied it to a general model of symmetric auctions.[1] Affiliation is a generalization of independence—see its definition in section 2—that was introduced through the appealing *positive dependence intuition*: "Roughly, this [affiliation] means that a high value of one bidder's estimate makes high values of the others' estimates more likely" (Milgrom and Weber (1982a, p.1096)).

Among other important results, Milgrom and Weber (1982a) were able to establish two key consequences of affiliation:

1. *Equilibrium existence*: there exists a pure strategy equilibrium in symmetric first-price auctions.[2] This is important because previous results required types to be independent.

2. *Revenue ranking*: under affiliation, the English and the second-price auction give higher expected revenue than the first-price auction (in self-explanatory

[1] In two previous papers, Milgrom (1981b) and Milgrom (1981a) presented results that used a particular version of the concept, under the name "monotone likelihood ratio property" (MLRP). It is also clear that Wilson (1969) and Wilson (1977) influenced the development of the affiliation idea. Nevertheless, the concept was fully developed and the term affiliation first appeared in Milgrom and Weber (1982a). See also Milgrom and Weber (1982b). When there is a density function, the property had been previously studied by statisticians under different names. Lehmann (1966)) calls it Positive Likelihood Ratio Dependence (PLRD), Karlin (1968) calls it Total Positivity of order 2 (TP_2) for the case of two variables or Multivariate Total Positivity of Order 2 (MTP_2) for the multivariate case.

[2] Milgrom and Weber (1982a) also proved the existence of equilibrium for second-price auctions with interdependent values. In our setup (private values), the second-price auction always has an equilibrium in weakly dominant pure strategies, which simply consists of bidding the private value.

symbols: $R_E \geqslant R_2 \geqslant R_1$).[3]

Taking together with the positive dependence intuition mentioned above, these two results suggest an economic interpretation in terms of comparative statics: when independence is relaxed in the direction of positive dependence, equilibrium is not a problem and the English auction (and second-price auction) gives higher revenue than the first-price auction. From an economic point of view, this comparative statics exercise is very interesting, since it clearly indicates what happens to the conclusion of the revenue equivalence theorem (RET) when one of its assumptions is relaxed from independence to positive dependence (or affiliation).[4]

We note, however, that affiliation *is not* positive dependence. Many different formalizations of positive dependence can be given; in section 3 we offer six of them. It is natural to ask how affiliation compares with these other concepts. Proposition 3.1 shows that affiliation is the strongest of all. This observation is important because the study of correlation or positive correlation in auctions has been almost exclusively restricted to affiliation, leading some researchers to equate the former with the later.[5] Proposition 3.1 clarifies that the distinction is potentially relevant and suggests a gap between the "lesson" that "positive dependence implies equilibrium existence and the revenue dominance of English auctions" and what we actually know.

This sets the stage to the main question in this paper: is it possible to extend the two main results mentioned above (equilibrium existence and revenue ranking) to other definitions of positive dependence? This question is important because positive dependence—rather than affiliation—is the intuitively appealing property. Once we realize that affiliation is not synonymous of positive dependence, one wonders how the above mentioned results hinge on the specific formulation introduced by Milgrom and Weber. Moreover, as we review in section 4, the intuitive explanation for both results seems to rely exclusively on positive dependence. Therefore, one should expect that weaker forms of positive dependence are sufficient for the result. Are they?

Section 4 answers this question in an interesting way. First, Theorems 4.1 and 4.2 show that both properties can be in fact generalized, at least in the context of

[3]For private value auctions, which is the focus of this paper, English and second-price auctions are equivalent, which implies $R_E = R_2$. See Milgrom and Weber (1982a).

[4]Besides independence, the RET requires other restrictive conditions, such as symmetry and risk neutrality. The revenue ranking of auctions is undetermined if all those assumptions are relaxed. Thus, the result is akin to a comparative statistics exercise: holding everything else fixed, what changes if independence is relaxed in the direction of positive dependence?

[5]Just to mention an example, consider the following: "The dynamic exchange of value information between bidders (...) is known to enhance revenue and efficiency in single item auctions with correlated values" Parkes (2006, p.42).

private value auctions. However, as we explain below, the generalization is not very significative; the required property is still restrictive. Can we go further in the generalization? Both theorems give counterexamples for further generalizations.

Our results are sharp, in the following sense. Proposition 3.1 ranks the seven properties in a strict rank of generality. Affiliation being Property VII, we generalize its implications for Property VI (decreasing inverse hazard rate), and show through counterexamples that the results fail for Property V (monotonicity in the first order stochastic sense). Curiously, exactly the same properties work for both results. It is even more interesting to note that monotonicity in the first order stochastic sense (Property V) is among the most used in economics. It has been so far unknown that it was not sufficient for implying equilibrium existence in auctions.[6]

The other sections of the paper are as follows. Section 2 describes the standard private value auction model. Section 5 briefly reviews the theoretical, experimental and empirical literature. As a complementary observation, section 6 documents that affiliation is restrictive in other senses. Specifically, Proposition 6.1 establishes that the set of continuous density functions that are *not* affiliated is open and dense in the set of all continuous density functions. In other words, the set of affiliated densities is meager. Section 7 concludes with some additional remarks, including direction for future research. An appendix collects the proofs.

2 Basic model and definitions

There are n bidders, $i = 1, ..., n$. Bidder i receives private information $t_i \in \left[\underline{t}, \bar{t}\right]$ which is the value of the object for himself. The usual notation $t = (t_i, t_{-i})$ = $(t_1, ..., t_n) \in \left[\underline{t}, \bar{t}\right]^n$ is adopted. The (private) values are distributed according to a pdf $f : \left[\underline{t}, \bar{t}\right]^n \to \mathbb{R}_+$ which is symmetric. That is, if $\pi : \{1, ..., n\} \to \{1, ..., n\}$ is a permutation, $f(t_1, ..., t_n) = f\left(t_{\pi(1)}, ..., t_{\pi(n)}\right)$. Let $\overline{f}(x) = \int f(x, t_{-i})\, dt_{-i}$ be a marginal of f. Our main interest is the case where f *is not* the product of its marginals, that is, the case where the types are dependent. We denote by $f(t_{-i} \mid t_i)$ the conditional density $f(t_i, t_{-i}) / \overline{f}(t_i)$.

After knowing his value, bidder i places a bid $b_i \in \mathbb{R}_+$. He receives the object if $b_i > \max_{j \neq i} b_j$. We consider both first and second-price auctions with *private values*. This means that the private information of each bidder (type) is also that bidder's value for the object. As Milgrom and Weber (1982a) show, second-price and English auctions are equivalent in the case of private values, as we assume here. In a first-price auction, if $b_i > \max_{j \neq i} b_j$, bidder i's utility is $u(t_i - b_i)$ and

[6] Even if the long period without a proof could suggest that this property was indeed not sufficient, only a clear result could dissipate the ineluctable doubt.

is $u(0) = 0$ if $b_i < \max_{j \neq i} b_j$. In a second-price auction, bidder i's utility is $u(t_i - \max_{j \neq i} b_j)$ if $b_i > \max_{j \neq i} b_j$ and $u(0) = 0$ if $b_i < \max_{j \neq i} b_j$. For both auctions, ties are randomly broken.

A pure strategy is a function $b : [0,1] \to \mathbb{R}_+$, which specifies the bid $b(t_i)$ for each type t_i. The interim payoff of bidder i, who bids β when his opponent $j \neq i$ follows $b : [0,1] \to \mathbb{R}_+$ is given by

$$\Pi_i(t_i, \beta, b(\cdot)) = u(t_i - \beta) F\left(b^{-1}(\beta) \mid t_i\right) = u(t_i - \beta) \int_{\underline{t}}^{b^{-1}(\beta)} f(t_j \mid t_i)\, dt_j,$$

if it is a first-price auction and

$$\Pi_i(t_i, \beta, b(\cdot)) = \int_{\underline{t}}^{b^{-1}(\beta)} u(t_i - b(t_j)) f(t_j \mid t_i)\, dt_j,$$

if it is a second-price auction.

We focus attention on symmetric monotonic pure strategy equilibrium (SMPSE), which is defined as $b(\cdot)$ such that $\Pi_i(t_i, b(t_i), b(\cdot)) \geqslant \Pi_i(t_i, \beta, b(\cdot))$ for all β and t_i. The usual definition requires this inequality to be true only for almost all t_i. This stronger definition creates no problems and makes some statements simpler, such as those about the differentiability and continuity of the equilibrium bidding function (otherwise, such properties should be qualified by the expression "almost everywhere"). Finally, under our assumptions, the second price auction always has a SMPSE in a weakly dominant strategy, which is $b(t_i) = t_i$.

By reparametrization, we may assume, without loss of generality, $\left[\underline{t}, \bar{t}\right] = [0,1]$. It is also useful to assume $n = 2$, but this is not necessary for most of the results. We also assume risk neutrality, i.e., $u(x) = x$. Thus, unless otherwise stated, the results will be presented under the following setup:

BASIC SETUP: *There are $n = 2$ risk neutrals bidders ($u(x) = x$), with private values distributed according to a symmetric density function $f : [0,1]^2 \to \mathbb{R}_+$.*

Affiliation is formally defined as follows.[7]

Definition 2.1 *The density function $f : \left[\underline{t}, \bar{t}\right]^n \to \mathbb{R}_+$ is affiliated if for any t and t' in $\left[\underline{t}, \bar{t}\right]^n$, we have $f(t) f(t') \leqslant f(t \wedge t') f(t \vee t')$, where $t \wedge t' = (\min\{t_1, t_1'\}, ..., \min\{t_n, t_n'\})$ and $t \vee t' = (\max\{t_1, t_1'\}, ..., \max\{t_n, t_n'\})$.*

[7] Affiliation is equivalent to MLRP in the particular case of two variables with density function. It is possible to define affiliation even if the joint distribution has no density function. See Milgrom and Weber (1982a).

It is useful to introduce the following notation: $\mathcal{D}$ will denote the set of all densities:

$$\mathcal{D} \equiv \{f : [0,1]^n \to \mathbb{R}_+ : \int_{[0,1]^n} f(t)dt = 1\}.$$

The set of all continuous densities will be denoted $\mathcal{C}$ and $\mathcal{A}$ will denote the set of affiliated (continuous or not) densities.

3 Affiliation and Positive dependence

Affiliation was introduced through the *positive dependence intuition*: "a high value of one bidder's estimate makes high values of the others' estimates more likely" (Milgrom and Weber (1982a, p. 1096)). This intuition is very appealing, because positive dependence describes a circumstance likely to happen in the real world. In fact, many authors introduce affiliation through this intuition or some of its variations.

Affiliation captures this intuition, as we illustrate in Figure 1, below. Affiliation requires that the product of weights at points (x', y') and (x, y) (where both values are high or both are low) be greater than the product of weights at (x, y') and (x', y) (where they are high and low, alternatively). In other words, the distribution puts more weight on the points in the diagonal than outside it.

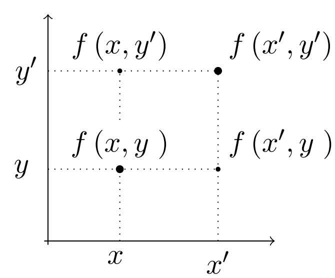

Figure 1 — The pdf f is affiliated if $x \leqslant x'$ and $y \leqslant y'$ imply $f\,(x, y')\, f\,(x', y) \leqslant f\,(x', y')\, f\,(x, y)$.

However, as long as we are interested in *positive dependence*, as this intuition suggests, affiliation is not the only definition available. In the statistical literature many concepts have been proposed to correspond to the notion of positive dependence. For simplicity, let us consider only the bivariate case and assume that the two real random variables X and Y have joint distribution F and strictly positive density function f. The following concepts are formalizations of the notion of positive dependence for X and Y:[8]

Property I — X and Y are positively correlated (PC) if $cov(X, Y) \geqslant 0$.

Property II — X and Y are said to be positively quadrant dependent (PQD) if for all non-decreasing functions g and h, $cov(g(X), h(Y)) \geqslant 0$.

Property III — The real random variables X and Y are said to be associated (As) if for all non-decreasing functions g and h, $cov(g(X, Y), h(X, Y)) \geqslant 0$.

Property IV — Y is said to be left-tail decreasing in X (denoted LTD($Y|X$)) if for all y, the function $x \mapsto \Pr[Y \leqslant y | X \leqslant x]$ is non-increasing in x. X and Y satisfy Property IV if LTD($Y|X$) and LTD($X|Y$).

Property V — Y is said to be positively regression dependent on X (denoted PRD($Y|X$)) if $\Pr[Y \leqslant y | X = x] = F(y|x)$ is non-increasing in x for all y. X and Y satisfy Property V if PRD($Y|X$) and PRD($X|Y$).[9]

Property VI — Y is said to be Inverse Hazard Rate Decreasing in X (denoted IHRD($Y|X$)) if $\frac{F(y|x)}{f(y|x)}$ is non-increasing in x for all y, where $f(y|x)$ is the pdf of Y conditional to X. X and Y satisfy Property VI if IHRD($Y|X$) and IHRD($X|Y$).

Since there are many alternative definitions of positive dependence, a natural question is: "How do such definitions compare with affiliation?" The following theorem provides the answer.[10]

Proposition 3.1 *Let affiliation be Property VII. Then,*

$$(VII) \Rightarrow (VI) \Rightarrow (V) \Rightarrow (IV) \Rightarrow (III) \Rightarrow (II) \Rightarrow (I),$$

and all implications are strict.

[8] Most of the concepts can be properly generalized to multivariate distributions. See, for example, Lehmann (1966) and Esary, Proschan, and Walkup (1967). The hypothesis of strictly positive density function is made only for simplicity.

[9] This property is also known as monotonicity in the first-order stochastic dominance sense.

[10] For this theorem, we used only seven concepts for simplicity. Yanagimoto (1972) defines more than thirty concepts of positive dependence and, again, affiliation is the most restrictive of all but one.

The main contribution of this theorem is around properties VII (affiliation), VI and V (first-order stochastic dominance), which are the most usual properties in economics. While Milgrom and Weber have proved $VII \Rightarrow VI$, we were not able to find a reference for the implication $VI \Rightarrow V$. The counterexamples $VI \nRightarrow VII$ and $V \nRightarrow VI$ are also new. These counterexamples are relevant because some confusion may arise with respect to the ranking of these properties.[11] Moreover, we are not aware of such counter-examples in the literature.

This theorem also sets the stage for our main problem. Since affiliation is just the strongest positive dependence property, could Milgrom and Weber choose a weaker property to get their results? This is the subject of the next section.

4 Main Results

Affiliation has been used in the proof of many results. These results can be classified in two groups: facts that are already true for the independent case (affiliation allows a generalization) and predictions that are qualitatively different from the case of independence. In this section, we will focus on one implication for each of these groups.

The first one is the existence of symmetric monotonic pure strategy equilibrium (SMPSE) for first price auctions, generalized from independence to affiliation. The second one is the revenue ranking of auctions: under affiliation, the English and the second-price auction give expected revenue at least as high as the first price auction (a fact that we denote by $R_2 \geqslant R_1$). This last result is in contrast with the case of independence, where the Revenue Equivalence Theorem (RET) implies the equality of the expected revenues $(R_2 = R_1)$.[12,13] Both implications were obtained by Milgrom and Weber (1982a) and I chose them because of their importance. The purpose of this section is to verify whether these implications (existence of SMPSE and $R_2 \geqslant R_1$) are true in a more general setting.

[11] A casual reader may think that Milgrom (1981a, Proposition 1) states that V is equivalent to VII and Riley (1988, Lemma 1) claims that a strict version of properties IV and V implies property VI. However, this is not the case—the mentioned results are formally correct. Proposition 3.1 helps to appreciate the subtle aspect of their claims.

[12] Since affiliation contains independence as a special case, the results can be *qualitatively* different, but must have a logic overlap.

[13] Both the revenue ranking under affiliation and the RET requires symmetry, risk neutrality and the same payoff by the lowest type of bidders.

4.1 Equilibrium existence

Is the existence of SMPSE true under other definitions of positive dependence (see section 3)? Theorem 4.1 below shows that the following property is sufficient:[14]

Property VI′ — The joint (symmetric) distribution of X and Y satisfy Property VI′ if for all x, x' and y in $[0,1]$,

$$x \geqslant y \geqslant x' \Rightarrow \frac{F(y|x')}{f(y|x')} \geqslant \frac{F(y|y)}{f(y|y)} \geqslant \frac{F(y|x)}{f(y|x)}.$$

It is easy to see that Property VI implies Property VI′ (under symmetry and full support). Thus, the question becomes whether or not it is possible to generalize the existence of SMPSE for Property V or even further.

If we define $\Pi(x,y) = (x - b(y)) F(y|x)$, where $b(\cdot)$ is a candidate for symmetric equilibrium,[15] then equilibrium existence is equivalent to $\Pi(x,x) \geqslant \Pi(x,y)$. Since $b(\cdot)$ is monotonic, one may conjecture that the monotonicity of $F(y|x)$ — as Property V assumes — may be sufficient for equilibrium existence, through some single crossing arguments (as in Athey (2001)). Since Property V is still a strong property of positive dependence, this conjecture may be considered reasonable. In fact, the reader may think that the following recent result by van Zandt and Vives (2007) actually *proves* that first-order stochastic dominance is sufficient for equilibrium existence in auctions:

Theorem (van Zandt and Vives, 2007): *Assume that for each player i:*

1. *the utility function is supermodular in the own player's action a_i, has increasing differences in (a_i, a_{-i}), and has increasing differences in (a_i, t); and*

2. *the beliefs mapping $p_i : T_i \to \mathcal{M}_i$ is increasing in the first-order stochastic dominance partial order.*

Then there exist a greatest and a least Bayesian Nash equilibrium, and each one is in monotone strategies.

[14] Motivated by an earlier version of this paper, Monteiro and Moreira (2006) obtained other generalizations of equilibrium existence for non-affiliated variables. Their results are not directly related to positive dependence properties.

[15] This candidate is increasing and unique, as we can show using standard arguments. See Maskin and Riley (1984) or de Castro (2012).

Despite these compelling reasons, the conjecture that Property V is sufficient for equilibrium existence in auctions is actually false; the following theorem clarifies that SMPSE existence does not generalize beyond Property VI'.[16]

Theorem 4.1 *If $f : [0,1]^2 \to \mathbb{R}$ satisfies Property VI$'$, there is a SMPSE. Nevertheless, Property V is not sufficient for the existence of SMPSE.*

This theorem shows that the rationale for existence of SMPSE existence for Property V do not survive a formalization of the result. Simply, Property V is not strong enough to control the equilibrium inequality $\Pi(x,x) \geqslant \Pi(x,y)$ for every pair of points (x,y).

4.2 Revenue ranking

The next implication—$R_2 \geqslant R_1$—is also an inequality, but it is an inequality over expected values, not specific realizations. For some realizations of the variables, the second-price auction can give less revenue than the first-price auction, but for the inequality $R_2 \geqslant R_1$ to be true is sufficient that the opposite happens on *average*. Since this is a statement about average cases, one could expect that the revenue ranking $R_2 \geqslant R_1$ would be stable across the cases of positive dependence.

There is yet another way of reaching the same conclusion: it is the intuition for the revenue ranking $R_2 \geqslant R_1$, which is a contribution of Klemperer (2004, p.48-9):

> [In a first-price auction, a] player with value $v + dv$ who makes the same bid as a player with a value of v will pay the same price as a player with a value of v when she wins, but because of affiliation she will expect to win a bit less often [than in the case of independence]. That is, her higher signal makes her think her competitors are also likely to have higher signals, which is bad for her expected profits.
>
> But things are even worse in a second-price affiliated private-values auction for the buyer. Not only does her probability of winning diminish, as in the first-price auction, but her costs per victory are higher. This is because affiliation implies that contingent on her winning the auction, the higher her value the higher expected second-highest value which is the price she has to pay. Because the person with the highest

[16] van Zandt and Vives (2007)'s main result does not apply because even simple auctions with 2 players and private-values do not satisfy one of their assumptions (increasing differences). In fact, if $t_i > a'_j > a'_i > a_j > a_i$ then $(t_i - a'_i)1_{[a'_i > a'_j]} - (t_i - a'_i)1_{[a'_i > a_j]} = -(t_i - a'_i) < 0$ while $(t_i - a_i)1_{[a_i > a'_j]} - (t_i - a_i)1_{[a_i > a_j]} = 0$, to the contrary of the increasing differences requirement.

value will win in either type of auction they are both equally efficient, and therefore the higher consumer surplus in first-price auction implies higher seller revenue in the second-price auction.

This intuition appeals mainly to the notion of positive dependence. Thus, the intuition should lead us to believe that the revenue ranking is still valid under weaker forms of positive dependence. Despite these intuitive arguments, however, the following theorem shows that the implication $R_2 \geqslant R_1$ is not robust for other definitions of positive dependence.

Theorem 4.2 *If f satisfies Property VI' (see definition above), then the second-price auction gives greater revenue than the first-price auction* $(R_2 \geqslant R_1)$. *Specifically, the revenue difference is given by*

$$n \int_0^1 \int_0^x b'(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx$$

where n is the number of players and $b(\cdot)$ is the first-price equilibrium bidding function, or by

$$n \int_0^1 \int_0^x \left[\int_0^y L(\alpha|y)\, d\alpha \right] \cdot \left[1 - \frac{F(y|x)}{f(y|x)} \cdot \frac{f(y|y)}{F(y|y)} \right] \cdot f(y|x)\, dy \cdot f(x)\, dx, \quad (1)$$

where $L(\alpha|t) = \exp\left[- \int_\alpha^t \frac{f(s|s)}{F(s|s)} ds \right]$.

More importantly, Property V is not *sufficient for this revenue ranking.*

These results suggest that affiliation's implications are not robust. This would not be a reason for concern, however, if affiliation were indeed typical. It is natural, therefore, to examine more closely the settings where affiliation could be expected to hold. The following section makes some comments about this issue.

5 Related literature

Few papers have pointed out restrictions or limitations to the implications of affiliation. Perry and Reny (1999) presented an example of a multi-unit auction where the linkage principle fails and the revenue ranking is reversed, even under affiliation. This result shows that revenue ranking is not robust when the number of objects increases from one to many. In contrast, one of our results shows that the revenue ranking is not robust even if we maintain the number of objects but allow for other kinds of dependences. Klemperer (2003) argues that, in real auctions, affiliation is not as important as asymmetry and collusion and illustrates his arguments with examples of the 3G auctions conducted in Europe in 2000–2001.

Nevertheless, much more has been written in accordance with the conclusions of affiliation. McMillan (1994, p.152) says that the auction theorists working as consultants to the FCC in spectrum auctions advocated for the adoption of the open auction using the linkage principle as one of the arguments: "Theory says, then, that the government can increase its revenue by publicizing any available information that affects the licensee's assessed value."[17] Milgrom (1989) emphasizes affiliation as the explanation for the predominance of the English auction over the first-price auction.

On the other hand, the experimental and empirical literature show an amazing lack of studies about whether affiliation holds or not. The empirical literature has tested affiliation's implication that the English auction gives higher revenue than the first-price auction, but there is no clear confirmation of this prediction. See Laffont (1997) for a survey of empirical literature on auctions. We are aware of only three papers proposing tests of affiliation: de Castro and Paarsch (2010), Jun, Pinkse, and Wan (2010) and Li and Zhang (2010). Those papers were motivated by an earlier version of this paper. The available experimental studies investigated only some of the implications of affiliation. See Kagel (1995) for a survey of this literature. See also section 7 below for suggestions of future work regarding this topic.

6 An observation about affiliation's typicality

The central topic of this paper is the robustness of affiliation's main implications. The fact that it is a restrictive form of positive dependence, established in Proposition 3.1, is just a motivation for this investigation. However, it is useful to document that affiliation is a restrictive assumption in other senses. In this section we show that the set of affiliated densities is small in the set of continuous densities. There are two ways to characterize a set as small: topological and measure-theoretic. Although it is possible to show that affiliation is restrictive in the measure-theoretic sense (see an earlier version of this paper, de Castro (2007)), here we limit ourselves to the topological result, which is simpler.

Recall that $\mathcal{C}$ denotes the set of continuous density functions $f : [0,1]^n \to \mathbb{R}_+$ and $\mathcal{A}$, the set of affiliated densities (continuous or not). Endow $\mathcal{C}$ with the standard topology for the set of continuous functions, that is, the topology defined by the

[17] Note that this is not necessarily their main argument, since they mentioned other advantages of the open auction, as "the bidders' ability to learn from other bids in the auction." McMillan (1994, p.152)

norm of the sup:

$$\|f\| = \sup_{x\in[0,1]^n} |f(x)| \,.$$

The following theorem shows that the set of continuous affiliated densities is small in the topological sense. The proofs of this and of all other results are given in the appendix.

Proposition 6.1 *The set of continuous affiliated density functions $\mathcal{C} \cap \mathcal{A}$ is meager.*[18] *More precisely, the set $\mathcal{C}\backslash\mathcal{A}$ is open and dense in $\mathcal{C}$.*

The proof of this theorem is given in the appendix, but the main idea is simple. To prove that $\mathcal{C}\backslash\mathcal{A}$ is open, we take a pdf $f \in \mathcal{C}\backslash\mathcal{A}$ which does not satisfy the affiliated inequality for some points $t,\ t' \in [0,1]^2$, that is, $f(t)\,f(t') > f(t \wedge t')\,f(t \vee t') + \eta$, for some $\eta > 0$. By using such η, we can show that for a function g sufficiently close to f, the above inequality is still valid, that is, $g(t)\,g(t') > g(t \wedge t')\,g(t \vee t')$ and thus is not affiliated. To prove that $\mathcal{C}\backslash\mathcal{A}$ is dense, we choose a small neighborhood V of a point $\hat{t} \in [0,1]^2$, such that for all $t \in V$, $f(t)$ is sufficiently close to $f\left(\hat{t}\right)$. This can be done because f is continuous. We then perturb the function in this neighborhood to maintain the failure of the affiliation inequality.

Maybe more instructive than the proof is the understanding of why the result is true: simply, affiliation requires an inequality to be satisfied everywhere (or almost everywhere). This is a strong requirement, and it is the source of affiliation's restrictiveness.

Although the restrictiveness of affiliation seems to be a "folk theorem," it was never stated or formally proven. Note that the conclusion of Proposition 6.1 may be sensitive to the space and norm considered, that is, the result might be false without the "right" statement, which indicates the value of formalizing it. Thus, Proposition 6.1 (together with the measure theoretical result proved in de Castro (2007)) fill this gap in the literature.

7 Final Remarks

As we observed in the introduction, there is no question that dependence is of fundamental importance in economics. It is also clear that we have experienced an

[18] A meager set (or set of first category) is the union of countably many nowhere dense sets, while a set is nowhere dense if its closure has an empty interior. Thus, the theorem says more than that $\mathcal{C} \cap \mathcal{A}$ is meager: $\mathcal{C} \cap \mathcal{A}$ is itself a nowhere dense set, according to the second claim in the theorem.

astonishing progress since affiliation was introduced as a foundation for the study of dependence by Milgrom and Weber (1982a). Almost thirty years later, a critical reassessment of the assumption seems overdue.

The intuitive appeal of affiliation through positive dependence is clear, yet as demonstrated in this paper, there are other ways to describe positive dependence that have different implications. This puts in perspective the reach of such implications.

Although we briefly reviewed the experimental and empirical literature, the scope of this paper was mainly theoretical. As we have seen, the respective literatures miss comprehensive studies about this topic.

Experimental studies could shed light on the actual distribution of values across individuals, controlling for the common knowledge. It would be very helpful to develop methods to determine the values that people attribute to objects in an auction and whether those values are correlated or not. With respect to econometrics, an obvious need is to develop methods to test the affiliation of bidders' values, controlling for unobserved heterogeneity. It would also be useful to develop techniques to describe the kind of dependence of the bids in real auctions. It would be very helpful to learn whether the kind of dependence is different across different markets and how these differences can be characterized. For instance, is there less correlation in Internet auctions, where the participants are consumers with almost no interaction, than in auctions where the participants are firms or professionals acting in the same industry? Yet another direction of research would be the development of econometric techniques to deal with dependence out of affiliation.[19]

It should be noted that the assessment presented in this paper is not a criticism of Milgrom and Weber (1982a)'s important results. On the contrary, Theorems 4.1 and 4.2 can be interpreted as saying that they have found not only a sufficient condition for their results, but also practically the most general one.[20]

On the other hand, this paper tries to deepen our understanding of affiliation, and how it relates to other aspects of positive dependence. Our results suggest that substantive progress in this field should require new approaches to dependence in economics.[21]

[19] Grid distributions can be useful for this task. See de Castro and Paarsch (2010).

[20] Although we generalize their results to property VI in the particular case of private values, this property is yet close to affiliation (property VII).

[21] A possible direction is offered by de Castro (2012), that introduces an assumption named richness to study correlation in Bayesian games. A particular case of richness is the set of grid distributions, which is useful for simulations and to test existence of symmetric monotone pure strategy equilibrium.

A Proofs

A.1 Proof of Proposition 3.1.

The proof of Proposition 3.1 is divided in two parts: the implications and the counterexamples.

A.1.1 Implications

It is obvious that $(III) \Rightarrow (II) \Rightarrow (I)$. The implication $(IV) \Rightarrow (III)$ is Theorem 4.3. of Esary, Proschan, and Walkup (1967). The implication $(V) \Rightarrow (IV)$ is proved by Tong (1980, p. 80). The implication $(VII) \Rightarrow (VI)$ is Lemma 1 of Milgrom and Weber (1982a). Thus, we need only to prove $(VI) \Rightarrow (V)$.

For this, assume that $H(y|x) \equiv \frac{f(y|x)}{F(y|x)}$ is non-decreasing in x for all y. Then, $H(y|x) = \partial_y [\ln F(y|x)]$ and we have

$$1 - \ln [F(y|x)] = \int_y^\infty H(s|x)\, ds \geqslant \int_y^\infty H(s|x')\, ds = 1 - \ln [F(y|x')],$$

if $x \geqslant x'$. Then, $\ln [F(y|x)] \leqslant \ln [F(y|x')]$, which implies that $F(y|x)$ is non-increasing in x for all y, as required by Property V.

A.1.2 Counterexamples

The counterexamples for each passage are given by Tong (1980, Chapter 5), except those involving Property (VI): $(V) \nRightarrow (VI)$, $(VI) \nRightarrow (VII)$. For the counterexample of $(V) \nRightarrow (VI)$, consider the following symmetric and continuous pdf defined on $[0,1]^2$:

$$f(x,y) = \frac{d}{1 + 4(y-x)^2}$$

where $d = [\arctan(2) - \ln(5)/4]^{-1}$ is the suitable constant for f to be a pdf. We have the marginal given by

$$f(y) = \frac{d}{2} [\arctan 2(1-y) + \arctan 2(y)]$$

so that we have, for $(x,y) \in [0,1]^2$:

$$f(x|y) = 2 \left[1 + 4(y-x)^2\right]^{-1} [\arctan 2(1-y) + \arctan 2(y)]^{-1},$$

$$F\left(x|y\right)=\frac{\left[\arctan 2\left(x-y\right)+\arctan 2\left(y\right)\right]}{\arctan 2\left(1-y\right)+\arctan 2\left(y\right)}$$

and

$$\frac{F\left(x|y\right)}{f\left(x|y\right)}=\frac{1}{2}\left[1+4\left(y-x\right)^{2}\right]\left[\arctan\left(2x-2y\right)+\arctan\left(2y\right)\right].$$

Observe that for $y'=0.91> y=0.9$ and $x=0.1$,

$$\frac{F\left(x|y'\right)}{f\left(x|y'\right)}>\frac{F\left(x|y\right)}{f\left(x|y\right)},$$

which violates Property (VI). On the other hand,

$$\begin{aligned}\partial_{y}\left[F\left(x|y\right)\right] &= \frac{\frac{2}{1+4y^{2}}-\frac{2}{1+4(x-y)^{2}}}{\arctan\left(2-2y\right)+\arctan\left(2y\right)}\\ &\quad -\frac{\left[\arctan\left(2x-2y\right)+\arctan\left(2y\right)\right]\left[\frac{2}{1+4y^{2}}-\frac{2}{1+4(1-y)^{2}}\right]}{\left[\arctan\left(2-2y\right)+\arctan\left(2y\right)\right]^{2}}\end{aligned}$$

In the considered range, the above expression is non-positive, so that Property (V) is satisfied. Then, $(V)\nRightarrow(VI)$.

Now we will establish that $(VI)\nRightarrow(VII)$. Fix an $\varepsilon<1/2$ and consider the symmetric density function over $[0,1]^{2}$:

$$f\left(x,y\right)=\begin{cases} k_{1}, & \text{if } x+y\leqslant 2-\varepsilon\\ k_{2}, & \text{otherwise}\end{cases}$$

where $k_{1}>1>k_{2}=2\left[1-k_{1}\left(1-\varepsilon^{2}/2\right)\right]/\varepsilon^{2}>0$ and $\varepsilon\in(0,1/2)$. For instance, we could choose $\varepsilon=1/3$, $k_{1}=19/18$ and $k_{2}=1/18$. The conditional density function is given by

$$f\left(y|x\right)=\begin{cases} 1, & \text{if } x\leqslant 1-\varepsilon\\ \frac{k_{1}}{k_{2}(x+\varepsilon-1)+k_{1}(2-\varepsilon-x)}, & \text{if } x>1-\varepsilon \text{ and if } y\leqslant 2-\varepsilon-x\\ \frac{k_{2}}{k_{2}(x+\varepsilon-1)+k_{1}(2-\varepsilon-x)}, & \text{otherwise}\end{cases}$$

and the conditional c.d.f. is given by:

$$F\left(y|x\right)=\begin{cases} 1, & \text{if } x\leqslant 1-\varepsilon\\ \frac{k_{1}y}{k_{2}(x+\varepsilon-1)+k_{1}(2-\varepsilon-x)}, & \text{if } x>1-\varepsilon \text{ and if } y\leqslant 2-\varepsilon-x\\ \frac{k_{2}(y+x+\varepsilon-2)+k_{1}(2-\varepsilon-x)}{k_{2}(x+\varepsilon-1)+k_{1}(2-\varepsilon-x)}, & \text{otherwise}\end{cases}$$

and

$$\frac{F(y|x)}{f(y|x)} = \begin{cases} 1, & \text{if } x \leqslant 1-\varepsilon \\ y, & \text{if } x > 1-\varepsilon \text{ and if } y \leqslant 2-\varepsilon-x \\ y+x+\varepsilon-2+k_1/k_2\,(2-\varepsilon-x)\,, & \text{otherwise} \end{cases}$$

Since $1 - k_1/k_2 < 0$, the above expression is non-increasing in x for all y, so that Property (VI) is satisfied. On the other hand, it is obvious that Property (VII) does not hold:

$$f(0.5,0.5)\, f\left(1-\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}\right) = k_2 k_1 < k_1^2 = f\left(0.5, 1-\frac{\varepsilon}{2}\right) f\left(0.5, 1-\frac{\varepsilon}{2}\right).$$

This shows that $(VI) \nRightarrow (VII)$.

A.2 Proof of Theorem 4.1.

The equilibrium existence follows from Milgrom and Weber (1982a)'s proof. The counterexample is in continuous values, but using the grid distributions proposed by de Castro (2012).[22] Consider the grid distribution $f : [0,1]^2 \to \mathbb{R}_+, f \in \mathcal{D}^4$ defined by:

$$f(x,y) = a_{mp} \text{ if } (x,y) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right],$$

for $m, p \in \{1,2,3,4\}$, where

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 2.0797 & 0.5505 & 1.4000 & 0.2296 \\ 0.5505 & 0.6965 & 0.5504 & 0.2439 \\ 1.4000 & 0.5504 & 2.3395 & 1.8158 \\ 0.2296 & 0.2439 & 1.8158 & 1.3040 \end{bmatrix}.$$

The definition of f at the zero measure set of points $\{(x,y) = \left(\frac{m}{k}, \frac{p}{k}\right) : m = 0$ or $p = 0\}$ is arbitrary. This distribution satisfies Property V but there does not exist a symmetric monotonic pure strategy equilibrium. These claims can be verified directly through tedious and lengthy calculations available upon request.

[22] I was unable to find examples with continuous variables without using grid distributions. I am grateful to Robert Wilson for pointing out an inconsistency in a previous example.

A.3 Proof of Theorem 4.2.

The dominant strategy for each bidder in the second-price auction is to bid his value: $b^2(t) = t$. Then, the expected payment by a bidder in the second-price auction, P^2, is given by:

$$\begin{aligned}
P^2 &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} y f(y|x)\, dy \cdot f(x)\, dx = \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} [y - b(y)]\, f(y|x)\, dy \cdot f(x)\, dx + \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b(y)\, f(y|x)\, dy \cdot f(x)\, dx,
\end{aligned}$$

where $b(\cdot)$ gives the equilibrium strategy for symmetric first-price auctions. Thus, the first integral can be substituted by $\int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y) \frac{F(y|y)}{f(y|y)} f(y|x)\, dy \cdot f(x)\, dx$, from the first-order condition: $b'(y) = [y - b(y)] \frac{f(y|y)}{F(y|y)}$. The last integral can be integrated by parts, to:

$$\begin{aligned}
& \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b(y)\, f(y|x)\, dy \cdot f(x)\, dx \\
=\ & \int_{[\underline{t},\bar{t}]} \left[b(x)\, F(x|x) - \int_{[\underline{t},x]} b'(y)\, F(y|x)\, dy \right] \cdot f(x)\, dx \\
=\ & \int_{[\underline{t},\bar{t}]} b(x)\, F(x|x) \cdot f(x)\, dx - \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y)\, F(y|x)\, dy \cdot f(x)\, dx
\end{aligned}$$

In the last line, the first integral is just the expected payment for the first-price auction, P^1. Thus, we have

$$\begin{aligned}
D &= P^2 - P^1 \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y)\, \frac{F(y|y)}{f(y|y)} f(y|x)\, dy \cdot f(x)\, dx \\
&\quad - \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y)\, F(y|x)\, dy \cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y) \left[\frac{F(y|y)}{f(y|y)} f(y|x) - F(y|x) \right] dy \cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx
\end{aligned}$$

Remember that $b(t) = \int_{[\underline{t},t]} \alpha dL(\alpha|t) = t - \int_{[\underline{t},t]} L(\alpha|t)\, d\alpha$, where $L(\alpha|t) = \exp\left[-\int_{\alpha}^{t} \frac{f(s|s)}{F(s|s)} ds\right]$. So, we have

$$\begin{aligned} b^{'}(y) &= 1 - L(y|y) - \int_{[\underline{t},y]} \partial_y L(\alpha|y)\, d\alpha \\ &= \frac{f(y|y)}{F(y|y)} \int_{[\underline{t},y]} L(\alpha|y)\, d\alpha. \end{aligned}$$

We conclude that

$$\begin{aligned} D &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} \frac{f(y|y)}{F(y|y)} \int_{[\underline{t},y]} L(\alpha|y)\, d\alpha \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)}\right] f(y|x)\, dy \cdot f(x)\, dx \\ &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} \left[\int_{[\underline{t},y]} L(\alpha|y)\, d\alpha\right] \cdot \left[1 - \frac{F(y|x)}{f(y|x)} \cdot \frac{f(y|y)}{F(y|y)}\right] \cdot f(y|x)\, dy \cdot f(x)\, dx, \end{aligned}$$

which is the desired expression if we multiply by the number n of players.

For the counterexample, consider the grid distribution $f : [0,1]^2 \to \mathbb{R}_+, f \in \mathcal{D}^4$ defined in the same fashion as in the proof of Theorem 4.1, by:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 2.7468 & 0.0803 & 0.1195 & 0.0696 \\ 0.0803 & 0.3200 & 0.5271 & 0.1224 \\ 0.1195 & 0.5271 & 1.7814 & 0.5650 \\ 0.0696 & 0.1224 & 0.5650 & 1.2705 \end{bmatrix}.$$

This distribution satisfies Property V (but not Property VI). Moreover, the first-price auction with this distribution has a SMPSE and a higher expected revenue than the correspondent second-price auction ($R_2 < R_1$). Again, these claims can be verified directly through tedious calculations.

A.4 Proof of Proposition 6.1 .

First, we prove that $\mathcal{C}\backslash\mathcal{A}$ is open. If $f \in \mathcal{C}\backslash\mathcal{A}$, then

$$f(x)\, f\left(x'\right) > f\left(x \wedge x'\right) f\left(x \vee x'\right),$$

for some $x, x' \in [0,1]^n$. Fix such x and x' and define $K = f(x) + f(x') + f(x \wedge x') + f(x \vee x') > 0$. Choose $\varepsilon > 0$ such that $2\varepsilon K < f(x) f(x') - f(x \wedge x') f(x \vee x')$ and let $B_\varepsilon(f)$ be the open ball with radius ε and center in f. Thus, if $g \in B_\varepsilon(f)$, $\|f - g\| < \varepsilon$, which implies $g(x) > f(x) - \varepsilon$, $g(x') >$

$f(x') - \varepsilon$, $g(x \wedge x') < f(x \wedge x') + \varepsilon$, $g(x \vee x') < f(x \vee x') + \varepsilon$, so that

$$\begin{aligned}
& g(x)\, g\left(x'\right) - g\left(x \wedge x'\right) g\left(x \vee x'\right) \\
> \;& [f(x) - \varepsilon]\left[f\left(x'\right) - \varepsilon\right] - \left[f\left(x \wedge x'\right) + \varepsilon\right]\left[f\left(x \vee x'\right) + \varepsilon\right] \\
= \;& f(x)\, f\left(x'\right) - f\left(x \wedge x'\right) f\left(x \vee x'\right) - \varepsilon\left[f(x) + f\left(x'\right) + f\left(x \wedge x'\right) + f\left(x \vee x'\right)\right] \\
= \;& f(x)\, f\left(x'\right) - f\left(x \wedge x'\right) f\left(x \vee x'\right) - \varepsilon K \\
> \;& \varepsilon K > 0,
\end{aligned}$$

which implies that $B_{\varepsilon}(f) \subset \mathcal{C} \backslash \mathcal{A}$, as we wanted to show.

Now, let us show that $\mathcal{C} \backslash \mathcal{A}$ is dense, that is, given $f \in \mathcal{C}$ and $\varepsilon > 0$, there exists $g \in B_{\varepsilon}(f) \cap \mathcal{C} \backslash \mathcal{A}$. Since $f \in \mathcal{C}$, it is uniformly continuous (because $[0,1]^n$ is compact), that is, given $\eta > 0$, there exists $\delta > 0$ such that $\|x - x'\|_{\mathbb{R}^n} < 2\delta$ implies $|f(x) - f(x')| < \eta$. Take $\eta = \varepsilon/4$ and the corresponding δ.

Choose $a \in (4\delta, 1 - 4\delta)$ and define x (x') by specifying that their first $\left\lfloor \frac{n}{2} \right\rfloor$ coordinates are equal to $a - \delta$ $(a + \delta)$ and the last ones to be equal to $a + \delta$ $(a - \delta)$. Thus, $x \wedge x' = (a - \delta, ..., a - \delta)$ and $x \vee x' = (a + \delta, ..., a + \delta)$. Let x_0 denote the vector $(a, ..., a)$. For $y = x$, x', $x \wedge x'$ or $x \vee x'$, we have: $|f(y) - f(x_0)| < \eta$. Let $\xi : (-1,1)^n \to \mathbb{R}$ be a smooth function that vanishes outside $\left(-\frac{\delta}{2}, \frac{\delta}{2}\right)^n$ and equals 1 in $\left(-\frac{\delta}{4}, \frac{\delta}{4}\right)^n$. Define the function g by

$$\begin{aligned}
g(y) &= f(y) + 2\eta\xi(y - x) + 2\eta\xi\left(y - x'\right) \\
&\quad -2\eta\xi\left(y - x \wedge x'\right) - 2\eta\xi\left(y - x \vee x'\right).
\end{aligned}$$

Observe that $\|g - f\| = 2\eta = \varepsilon/2$, that is, $g \in B_{\varepsilon}(f)$. In fact, $g \in B_{\varepsilon}(f) \cap \mathcal{C} \backslash A$, because

$$\begin{aligned}
g(x) &= f(x) + 2\eta > f(x_0) + \eta; \\
g\left(x'\right) &= f(x) + 2\eta > f(x_0) + \eta; \\
g\left(x \wedge x'\right) &= f\left(x \wedge x'\right) - 2\eta < f(x_0) - \eta; \\
g\left(x \vee x'\right) &= f\left(x \vee x'\right) - 2\eta < f(x_0) - \eta,
\end{aligned}$$

which implies

$$\begin{aligned}
& g(x)\, g\left(x'\right) - g\left(x \wedge x'\right) g\left(x \vee x'\right) \\
> \;& [f(x_0) + \eta]^2 - [f(x_0) - \eta]^2 \\
= \;& 4\eta > 0,
\end{aligned}$$

as we wanted to show. ∎

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