Economic Theory (2008) 35:407–440
DOI 10.1007/s00199-007-0249-0

RESEARCH ARTICLE

Non-monotoniticies and the all-pay auction tie-breaking rule

Aloisio Araujo · Luciano I. de Castro · Humberto Moreira

Received: 9 October 2006 / Revised: 11 April 2007 / Published online: 14 July 2007

Abstract Discontinuous games, such as auctions, may require special tie-breaking rules to guarantee equilibrium existence. The best results available ensure equilibrium existence only in mixed strategy with endogenously defined tie-breaking rules and communication of private information. We show that an all-pay auction tie-breaking rule is sufficient for the existence of pure strategy equilibrium in a class of auctions. The rule is explicitly defined and does not require communication of private information. We also characterize when special tie-breaking rules are really needed.

Keywords Auctions · Pure strategy equilibria · Non-monotonic bidding functions · Tie-breaking rules · Necessary and sufficient conditions for equilibrium · Multidimensional types

JEL Classification Numbers C62 · C72 · D44 · D82

A. Araujo
Instituto Nacional de Matemática Pura e Aplicada (IMPA) and Graduate School of Economics, Getulio Vargas Foundation (EPGE/FGV), Rio de Janeiro, Brazil
e-mail: aloisio@impa.br

L. I. de Castro (✉)
Universidad Carlos III de Madrid, Madrid, Spain
e-mail: decastro.luciano@gmail.com

H. Moreira
Graduate School of Economics, Getulio Vargas Foundation (EPGE/FGV), Rio de Janeiro, Brazil
e-mail: humberto@fgv.br

L. I. de Castro
University of Illinois at Urbana-Champaign, Champaign, IL, USA

1 Introduction

The existence of equilibrium for discontinuous games depends on special definition of the payoff at discontinuities. This was first pointed out by Simon and Zame (1990) for games of complete information, and by Jackson et al. (2002), henceforth JSSZ, for games of incomplete information.

Both papers offer interesting examples where there is no equilibrium if the payoff at discontinuities is defined in the usual manner. Indeed, for most of the games there is a standard way to define the payoff at discontinuities. For auction games, discontinuities occur at ties (with positive probability) and it is usual to define the payoff at such points through the *standard tie-breaking rule,* which consists in randomly splitting the object among the tying bidders.

Motivated by examples, both papers offer a solution concept where the payoff at the discontinuities is defined as the limit point of a sequence of payoffs. Thus, the payoff is "endogenously defined" at the discontinuities and impossible to characterize, since there is no available criterion to select a priori which accumulation point is the correct one.

In the standard definition of a game, the payoffs of all players for every profile of actions are specified. Thus, their solution requires the definiton of the game to be weaker (or more incomplete) than the usual one. This suggests the importance of understanding when special rules are really needed and obtaining the explicit payoff at discontinuities that ensure equilibrium existence. For auctions, this means explicit tie-breaking rules.

Simon and Zame (1990) write: "...the same sharing rule should not be expected to work for all problems. Indeed, even within the same problem, different sharing rules may be appropriate at different points" (p. 862). Nevertheless, we show that it is possible to define an explicit tie-breaking rule that works for a class of auctions.

Specifically, we show that the *all-pay auction tie-breaking rule* is sufficient for the existence of pure strategy equilibria in a set of symmetric auctions for which the best results available (JSSZ) ensure only the existence in mixed strategies with an endogenously defined tie-breaking rule.[1] Furthermore, we completely characterize when the standard tie-breaking rule is enough.

The rule is: if there is a tie, conduct an all-pay auction among the tying bidders. If there is still a tie, split the object randomly.[2] We show that ties occur with zero probability in the second round, which ensures equilibrium existence in pure strategies. Moreover, the all-pay auction tie-breaking rule does not require communication of private information, as JSSZ's solution does.

Besides the definition of an explicit tie-breaking rule, we characterize the situations where they are really needed. We illustrate this characterization with JSSZ's Example 1. This is a standard first-price auction except for the fact that the bidder's payoff is decreasing in the opponents' types, although increasing in her own (this is

[1] In this paper, we focus only on pure strategy equilibria.

[2] The all-pay auction tie-breaking rule is similar to Maskin and Riley's second-price auction tie-breaking rule (Maskin and Riley 2000), where the tie is broken through a second-price auction, instead of an all-pay auction, as in our case.

also similar to example 3 of Maskin and Riley 2000). The need for special tie-breaking rules is thus related to non-monotonicities. In fact, it turns out that non-monotonicities of the payoff function are crucial to the need of special tie-breaking rules—when types and values do not have atoms.[3]

We argue that non-monotonicities may arise in meaningful economic situations (see Sect. 4). Thus, the point goes beyond the technical interest in equilibrium existence. Our results suggest that the all-pay auction is a good mechanism to "rank" the bidders' information as required for equilibrium existence (see the discussion following Theorem 3, in Sect. 3.2).

Nevertheless, not all non-monotonic utility functions require special tie-breaking rules. As stated above, our results allow us to characterize the set of symmetric auctions for which the standard tie-breaking rule is sufficient to ensure equilibrium existence. It turns out that this strictly contains the set of auctions with non-decreasing interdependent values, but may also include situations like example 1 of JSSZ. In addition, we will show that there are examples close to theirs that do not have equilibrium with the standard tie-breaking rule.[4]

The paper is organized as follows: in Sect. 2 we present the model. Section 3 gives the main results for unidimensional auctions with monotonic equilibria: the all-pay auction tie-breaking rule is introduced and equilibrium existence is proved. In Sect. 4 we discuss why non-monotonicities can be of interest. In Sect. 5 we extend some results to multidimensional auctions and non-monotonic value and bidding functions. Section 6 concludes with a discussion on the related literature. An appendix collects the proofs.

2 The model

There are N bidders in an auction of a single object.[5] Player i ($i = 1, \ldots, N$) has private information $t_i \in S$, where (S, Σ, μ) is a non-atomic probabilistic space.[6] Then, bidder i chooses a bid $b_i \in B \equiv \{b_{OUT}\} \cup [b_{\min}, +\infty)$, where $b_{\min} > b_{OUT}$ is the minimal valid bid. If $b_i = b_{OUT}$, bidder i does not participate in the auction and gets 0.

Let $\left(S^N, \Sigma^N, \mu^N\right)$ be the product of N independent copies of (S, Σ, μ), defined in the usual way. Let $t = (t_i, t_{-i}) \in S^N$ be the profile of signals and $b = (b_i, b_{-i})$ the profile of submitted bids. The cutoff that determines the winning and losing events for bidder i is

$$b_{(-i)} \equiv \max\{b_{\min}, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_N\}.$$

i.e., the bidder i receives the object if $b_i > b_{(-i)}$ and none if $b_i < b_{(-i)}$. If the tie-breaking rule is not explicitly mentioned, we assume that ties ($b_i = b_{(-i)}$) are broken

3 For the case of first-price auctions with atoms in the distribution of private values, see Monteiro (2004).

4 The Example 2 in Sect. 3.2 embeds example 1 of JSSZ in a class of parametrized examples. By "a close example" we mean a neighborhood of the parameters that specify their example.

5 The model can be easily extended to $L < N$ homogenous objects, if each bidder's demand is unitary.

6 In Sect. 3 we consider only the case $S = [0, 1] \subset \mathbb{R}$ and $N = 2$. In Sect. 5 we comment on how these assumptions can be relaxed.

by the standard tie-breaking rule, that is, the object is randomly divided among the tying bidders. More specifically, the payoff of bidder i is given by

$$u_i(t,b) = \begin{cases} v(t_i, t_{-i}) - p^W\left(b_i, b_{(-i)}\right), & \text{if } b_i > b_{(-i)} \\ -p^L\left(b_i, b_{(-i)}\right), & \text{if } b_i < b_{(-i)} \\ \frac{v(t_i, t_{-i}) - b_i}{m(b)}, & \text{if } b_i = b_{(-i)} \end{cases}$$

where $v(t_i, t_{-i})$ is the value of the object for bidder i, p^W and p^L are the payments made in the events of winning and losing, respectively, and $m(b)$ is the number of tying bidders.

Our setting is given by the following assumptions:

Assumption 1 Types are independently distributed in $\left(S^N, \Sigma^N, \mu^N\right)$, where μ is the marginal distribution for the type of each bidder. Players are risk-neutral and the value of the object for player i is given by $v(t_i, t_{-i})$, where the function $v : S \times S^{N-1} \to \mathbb{R}_+$ is measurable, its range is the compact interval $\left[\underline{v}, \overline{v}\right] \subset \mathbb{R}_+$ and it is symmetric in the last $N-1$ arguments, that is, if t'_{-i} is a permutation of t_{-i}, $v\left(t_i, t'_{-i}\right) = v(t_i, t_{-i})$. Moreover, $\mu^N \circ v^{-1}$ is absolutely continuous with respect to the Lebesgue measure.[7]

The most restrictive requirement of Assumption 1 is symmetry, although independence is also restrictive. However, available results on pure strategy equilibrium existence are restricted to independency or affiliation. On the other hand, note that Assumption 1 does not require any kind of monotonicity and, thus, generalizes, in this direction, assumptions usually required in auction models.

The specific auction format is determined by p^W and p^L. We will alternatively consider two cases. The first one, embodied in Assumption 2(i) below, covers first-price auctions and all-pay auctions. The second case, defined by Assumption 2(ii), covers second-price auctions, among other formats.

Assumption 2 For $j = W$ or L , $p^j(\cdot,\cdot) \geqslant 0$, $p^j(b_{OUT}, \cdot) = 0$, $\partial_1 p^j \geqslant 0$, $p^j(\cdot, b_{OUT}) = p^j(\cdot, b_{\min})$, p^j is differentiable over $(b_{\min}, \infty) \times (b_{\min}, \infty)$, $p^L(b_{\min}, b) = p^L\left(b_{\min}, b'\right)$ fr all b and b' and one of the two conditions below is satisfied:

(i) $\partial_1 p^W(\cdot) > 0$ or $\partial_1 p^L(\cdot) > 0$;
(ii) $\partial_1 p^W = \partial_1 p^L \equiv 0$ and $\partial_2\left(p^W - p^L\right) > 0$.

Assumption 2 allows us to cover virtually all kinds of standard single-object or multi-unit auctions with unitary demands and encompasses the use of entry fee. Some important examples are:

(A) All-pay auctions: $p^W\left(b_i, b_{(-i)}\right) = b_i$ and $p^L\left(b_i, b_{(-i)}\right) = b_i$.
(F) First-price auctions: $p^W\left(b_i, b_{(-i)}\right) = b_i$ and $p^L\left(b_i, b_{(-i)}\right) = 0$.

[7] That is, if $C \subset \mathbb{R}$ has zero Lebesgue measure, then $\mu^N\{\left(t_i, t_{-i}\right) \in S^N : v\left(t_i, t_{-i}\right) \in C\} = 0$. Observe that this implies that μ^N is non-atomic. Thus, this part of the assumption is just a generalization of the usual assumption for monotonic auctions, where v is strictly increasing with t_i and μ^N is non-atomic. Indeed, these conditions imply the absolute continuity of $\mu^N \circ v^{-1}$.

(S) Second-price auctions: $p^W\left(b_i, b_{(-i)}\right) = b_{(-i)}$ and $p^L\left(b_i, b_{(-i)}\right) = 0$.
(W) War of attrition: $p^W\left(b_i, b_{(-i)}\right) = b_{(-i)}$ and $p^L\left(b_i, b_{(-i)}\right) = b_i$.

An active reserve price, that is, $b_{\min}$, which excludes some bidders, is dealt with in the Appendix. However, for a simple statement of the results, we restrict our exposition to the case where the reserve price is not active, as summarized by the following assumption:

Assumption 3 v, p^W, p^L and $b_{\min}$ are such that no bidder plays b_{OUT}, that is, no bidder prefers to stay out of the auction.

We denote the auction described above by $\mathcal{A} = (S, \Sigma, \mu, N, v)$. Observe that we are considering only symmetric auctions. Thus, throughout the paper, when we refer to a strategy or to a profile of strategies, we always mean symmetric pure strategies.

3 Pure strategy monotonic equilibria for non-monotonic auctions

Our first aim is to characterize the conditions under which there exist symmetric monotonic equilibria in a setting where the payoff functions are not necessarily monotonic. For all results of this section, we consider an auction $\mathcal{A} = (S, \Sigma, \mu, N, v)$ satisfying Assumptions 1, 2 and 3 and such that $N = 2$ (there are only 2 players) and $S = [0, 1] \subset \mathbb{R}$.[8]

Let us denote by $p(\beta, b)$ the expected payment of a bidder who plays a bid β, when the opponent is playing an increasing strategy $b : [0, 1] \to \mathbb{R}$ in the auction $\mathcal{A}$, that is,

$$p(\beta, b) \equiv \int_0^{b^{-1}(\beta)} p^W(\beta, b(\alpha))\, d\alpha + \int_{b^{-1}(\beta)}^{1} p^L(\beta, b(\alpha))\, d\alpha.$$

The first useful result that we derive is the Revenue Equivalence Theorem for our setting.

Proposition 1 (Revenue Equivalence Theorem) *Let $b : [0, 1] \to [b_{\min}, +\infty)$ be a strictly increasing symmetric equilibrium of $\mathcal{A}$. If v is continuous, then*

$$p(b(y), b) = \int_0^{y} v(\alpha, \alpha)\, d\alpha. \tag{1}$$

Proof It is a direct consequence of Proposition 7 in the Appendix. □

We also have the following:

---

[8] All the results of this section can be extended to $N > 2$ players, as we explain in Sect. 5, but we keep $N = 2$ to simplify notation. In that section we also discuss the extensions to multidimensional type spaces.

Proposition 2 *Assume that v is continuous. If $b : [0, 1] \to [b_{\min}, +\infty)$ is a strictly increasing symmetric equilibrium of $\mathcal{A}$, then Assumption 2(i) implies that b is differentiable and*

$$b'(x) = \frac{v(x,x) - p^W(b(x), b(x)) + p^L(b(x), b(x))}{E\left[\partial_1 p^W(b(x), b(\cdot))\, 1_{[b(x)>b(\cdot)]} + \partial_1 p^L(b(x), b(\cdot))\, 1_{[b(x)<b(\cdot)]}\right]}, \quad (2)$$

while Assumption 2(ii) implies that b is continuous and

$$v(x,x) - p^W(b(x), b(x)) + p^L(b(x), b(x)) = 0. \quad (3)$$

Proof The first part is a consequence of Proposition 4, while the second part is a consequence of Proposition 5 in the Appendix. □

For the four previously mentioned formats, the function b that satisfies (1) is given by:

$$\text{(All-pay auction)} \quad b(x) = \int_0^x v(\alpha, \alpha)\, d\alpha \quad (4)$$

$$\text{(First-price auction)} \quad b(x) = \frac{1}{x}\int_0^x v(\alpha, \alpha)\, d\alpha \quad (5)$$

$$\text{(Second-price auction)} \quad b(x) = v(x, x) \quad (6)$$

$$\text{(War of attrition)} \quad b(x) = \int_0^x \frac{v(\alpha, \alpha)}{1-\alpha} d\alpha. \quad (7)$$

It is easy to see that each of these functions is increasing if v is non-decreasing in both arguments. Nevertheless, since we do not assume such monotonicity, this is not necessarily true in our setting. This observation will be important below.

Recall that Assumption 2(ii) requires that $\partial_1 p^W = \partial_1 p^L \equiv 0$ and $\partial_2 \left(p^W - p^L\right) > 0$. Thus, if v is continuous, as required by Proposition 2, then (3) always has a solution, by the Intermediate Value Theorem. However, the existence of b does not imply its monotonicity. Under Assumption 2(i), we might impose extra conditions to ensure the existence of a solution to (2), i.e., the existence of b satisfying (1).

The next theorem provides an extra necessary weak monotonicity condition on v for the existence of a monotonic equilibrium. It turns out that this condition is sufficient to guarantee that an increasing bidding function b satisfying the payment expression (1) is an equilibrium.

Theorem 1 (Necessary and Sufficient Conditions for Equilibrium) *Suppose that v is continuous. There exists a symmetric monotonic equilibrium without ties with positive probability if and only if*

(i) *there exists a strictly increasing continuous function b that satisfies the payment expression (1);*

(ii) *for all* $(x, y) \in [0, 1] \times [0, 1]$,

$$\int_y^x [v(x, \alpha) - v(\alpha, \alpha)]\, d\alpha \geqslant 0. \tag{8}$$

In the affirmative case, the function b in item (i) is an equilibrium of $\mathcal{A}$. If v is not continuous, the above conditions are still sufficient and (ii) is necessary.

Proof Proposition 1 shows that (i) is necessary. The necessity of (ii) and the sufficiency part are shown by Corollary 9 in the Appendix. □

The condition (ii) of Theorem 1 is a weak monotonicity condition since (8) is satisfied when $[v(x, z) - v(z, z)](x - z) \geq 0$, for all $x, z \in [0, 1]$. In particular, $x \mapsto v(x, z)$ increasing for all $z \in [0, 1]$ is sufficient for (8).

The condition that b satisfying (1) is increasing is a condition on the primitives of the problem and it is straightforward to check for specific auction formats. For instance, in the case of a first-price auction we only have to check whether the function given by (5) is increasing. The same is valid for all-pay auctions (4), second-price auctions (6) and war of attrition (7). The verification is not straightforward only when (2) or (3) have no close form solution.

Section 3.2 will treat cases where b satisfying (1) is not increasing.

3.1 An example

In Example 1 below, condition (8) is not satisfied and there is no monotonic equilibrium. In the Appendix we reparametrize the types of Example 1 such that Theorem 1 is satisfied for the new types. The equilibrium bidding function is not monotonic in the original types.

Example 1 Consider a symmetric first-price auction between two bidders with independent and uniformly distributed types in [0, 1]. The payoff function is

$$v(x, y) = \frac{9(x + 1)(2y - x + 1)}{8}.$$

It is easy to verify that b given by (5) is increasing, but the necessary condition (8) is not satisfied, because

$$\int_y^x [v(x, \alpha) - v(\alpha, \alpha)]\, d\alpha = -\frac{3(x - y)^3}{8} < 0$$

for $x > y$. By Theorem 1, there is no monotonic equilibrium. However, in the Appendix we show that the following continuous and bell-shaped bidding function is an equilibrium:

$$b(x) = \begin{cases} \frac{3(12+3x-2x^2)}{16}, & \text{if } x \in \left[0, \frac{1}{2}\right] \\ \frac{3(13+x-2x^2)}{16}, & \text{if } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Example 1 shows that sometimes only non-monotonic equilibria are possible. Thus, natural questions arise: when only increasing equilibria exist and when equilibria exist at all (possibly with non-monotonic strategies). The latter question will be tackled in Sect. 5, while the former is treated in the sequel.

3.2 Monotonic equilibria and the all-pay auction tie-breaking rule

The traditional auction theory usually requires the monotonicity of $v(x, y)$ in both arguments. In what follows we make a weaker assumption:

Assumption 4 For all $y \in S$, $x \mapsto v(x, y)$ is increasing.

For the results of this subsection, we assume that Assumption 4 also holds. Since Assumption 4 trivially implies (8), the sufficiency for equilibrium existence in the theorem that follows is an immediate consequence of Theorem 1.

Theorem 2 (Existence of Equilibrium) *b is an equilibrium of $\mathcal{A}$ without ties* (*with positive probability*) *if and only if b is increasing and satisfies the payment expression* (1).

Example 1 does not satisfy Assumption 4 and has a non-monotonic equilibrium obtained through a reparametrization of types. Under Assumption 4, such repametrizations do not work. On the other hand, it is important to emphasize that Assumption 4 does not imply that b is increasing, as the following example shows.

Example 2 (Example 1 of JSSZ. See also Jackson et al. 2004) Let $v(x, y) = \alpha + x + \beta y$ be a utility function clearly satisfying Assumption 4. The first-price bidding function given by (5)

$$b(x) = \frac{1}{x} \int_0^x v(z, z)\, dz = \frac{1}{x}\left[\alpha x + \frac{(1+\beta)\, x^2}{2}\right] = \alpha + \frac{(1+\beta)\, x}{2}$$

is increasing only if $\beta > -1$. Thus, there is an equilibrium without ties if and only if $\beta > -1$, provided that $\alpha \geqslant \max\{0, -(1+\beta)/2, -\beta\}$ (otherwise, there would exist negative values).

Example 2 is used by JSSZ to show that the equilibrium may fail to exist under the standard tie-breaking rule. Building on this example, they propose an endogenously defined tie-breaking rule to ensure equilibrium existence in general.

Let us consider instead the exogenous *all-pay auction tie-breaking rule*: if a tie occurs, conduct an all-pay auction among the tying bidders. If another tie occurs, split the object evenly. The payments of the tying bidders are made according to the second-stage bids. Our all-pay auction tie-breaking rule has the role of the second-price auction tie-breaking rule used by Maskin and Riley (2000).

The next theorem shows that the all-pay auction tie-breaking rule always ensures equilibrium existence for the auctions we are considering. It is also important to note that the all-pay auction tie-breaking rule does not require an announcement of types.[9]

Theorem 3 (Equilibrium Existence with Ties) *Assume that the all-pay auction tie-breaking rule is adopted. If there is b that satisfies the payment expression (1), then there exists a pure strategy equilibrium.*

Proof This follows from the proof of Theorem 6. □

The function b satisfying the statement of Theorem 3 is an equilibrium only if it is increasing. If not, we can modify b into a non-decreasing bidding function with a positive-probability set of tying types.

One important ingredient in the proof of Theorem 3 is that the bidding function of an all-pay auction

$$b(x) = \int_0^x v(\alpha, \alpha)\, d\alpha$$

is always increasing (since v is positive by Assumption 1) and gives exactly the expected payment (1). Any other auction that has an increasing bidding equilibrium, like the all-pay auction or war of attrition, can be used as the tie-breaking mechanism.

Example 2 (*continuation*) For $\beta \leqslant -1$, the equilibrium of Example 2 is given by a constant bidding function $b^1(x) = \bar{b}$ for the first-price auction, where $\bar{b} \in \left[\alpha + \frac{1+\beta}{2}, \alpha\right]$, and the bidding function

$$b^2(x) = \alpha x + \frac{1+\beta}{2} x^2$$

for the second-phase auction (the all-pay auction that breaks the ties). □

Tournaments

When b is not increasing, types are not "correctly" ordered and b fails to conveniently reveal the bidder's information. The tie-breaking rule plays exactly the role of sorting the types. Thus, Theorem 3 shows that all-pay and war-of-attrition auctions are better-revealing information mechanisms than first-price and second-price auctions.

[9] This does not contradict the example given by Jackson et al. (2004) on the non-existence of equilibrium with type-independent tie-breaking rule. Although related, the two concepts are distinct: our rule is type-dependent in their sense because the final outcome depends on the players' types through the outcome of the second-stage auction. There is also uncertainty on the number of available objects in their modified example, while we consider only standard auctions, that is, with a fixed number of objects.

An important example of all-pay auctions is a tournament. Our previous theorem thus gives a justification for the practice of tournaments. Consider, for instance, the case of research contest among researchers who have a vector of unobserved characteristics, such as technical capabilities, discipline, honesty, creativity, etc. In such cases the object value is usually a very intricate function of these characteristics and tournaments (all-pay auctions) can perform better the task of revealing this multidimensional information. Thus, tournaments are likely in situations where the determination of the best competitor is more complex.

Standard explanations for the use of tournaments in research contest also appeal to the role of information. Taylor (1995, p. 872) says that: "Contracting for research is often infeasible because research inputs are unobservable and research outcomes cannot be verified by a court". Our point is somewhat different but related to this. The novelty is the comparison between auction mechanisms with respect to information revelation.

That all-pay auctions are better mechanisms to reveal information is not completely new. Fullerton and McAfee (1999) observe that the all-pay auction implies the existence of an increasing equilibrium in a setting where the second-price auction does not. They analyze the auction for the right to compete in a tournament—see Sect. 4 for further explanation. However, their auctions' model is a particular case of ours. Thus, Theorem 3 extends their basic intuition and shows that the property of better-revealing information can be used to define an effective tie-breaking rule that does not require an announcement of types.

Multiplicity of equilibria

An interesting corollary of Theorem 3 is the possibility of multiple equilibria when b is not increasing, even under Assumption 4. There are two sources for this multiplicity. The first is due to the all-pay auction tie-breaking rule per se, since many bid levels can be chosen as the tying bid. This is shown in Fig. 1: any level b_0 between z_0 and z_1 can be chosen in the interval where b is decreasing and gives the same expected payment and payoff to each bidder in the auction.

The second source is the fact that the tie-breaking rule is not unique in general. For instance, for some specifications of v there are cases where b is decreasing with many tie-breaking rules that ensure equilibrium existence (see Example 1 of JSSZ). However, these tie-breaking rules may result in different expected revenues, whereas the all-pay auction tie-breaking rule always preserves the Revenue Equivalence Theorem.

4 Non-monotonicities in multidimensional auctions

In JSSZ the action and type spaces are compact and metric and no monotonicity condition is imposed on the utility functions. Thus, the need for special tie-breaking rules has to be understood in a more general setup. Nevertheless, theoretical generality is not the only motivation. Even if we restrict ourselves to auctions, there are meaningful and interesting situations where the usual monotonic assumptions are too restrictive.

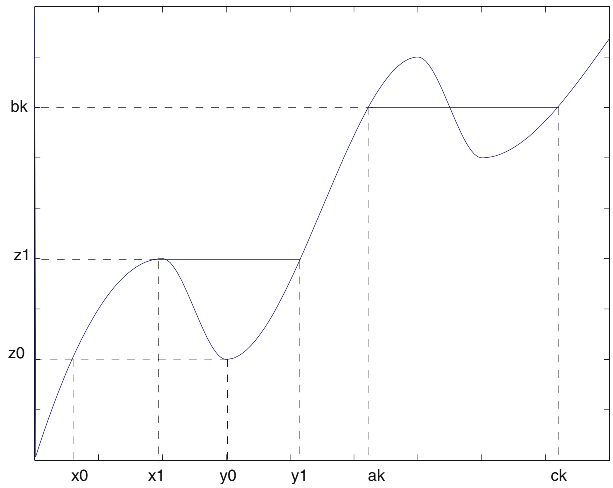

Fig. 1 Non-monotonic bidding function

An interesting example occurs when the private information is not directly related to the object value. For instance, the private information can be the financial constraints of the bidder. In this case, the utility functions are not necessarily increasing with the private information. Indeed, Zheng (2001) considers a case where the private information is the financial capacity of the bidder and non-monotonic equilibria can arise (see his Remark 3.2, p. 157).

The possibility of private information not directly related to value is, per se, a motivation to consider models where the utility function is not necessarily monotonic. But, even if it is directly related to the value, non-monotonicities can also arise. Consider the following:

Example 3 (Job Market) Let us model the job market for a potential manager as an auction between two competing firms where the object is the job contract. It is natural to assume that managers have a multidimensional vector of characteristics, $m = (m^1, \ldots, m^k)$. For simplicity, we consider only one-dimensional characteristics. The qualification appraisal of the job candidate is private information of the competing firms. Each firm has just one position with a specific characteristic requirement. In general, for specific jobs firms have a desirable level of a specific characteristic. To give a concrete example, if the characteristic is experience, an experienced candidate can be rejected for a junior position.[10] Therefore, let d_i be such a desirable level. Thus, the utility of the firms in this auction is modeled as

[10] Non-monotonicities seem to be a general feature of job markets. For instance, if the characteristic is the ability to communicate and the position is a librarian, the desirable level of this ability is much lower than if the position is for a salesman. An investment bank may want a sufficiently (but not exaggeratedly) high level of risk-loving or audacity, while a family business may want a much lower level. Even efficiency or qualification can be desirable on different levels. Sometimes, the rejection of a candidate is explained by "over-qualification".

$$v_i(t_i, t_{-i}) = a\frac{t_1 + t_2}{2} - b\left(\frac{t_1 + t_2}{2} - d_i\right)^2$$

where a is the weight of the managers' characteristic and $b > 0$ is the penalty of the distance from the desired level d_i. It is clear that this utility may be non-monotonic in the types. For some values of the constants, it is also possible that all equilibria are non-mononotic (a proof of this last claim is available upon request).

Another example where non-monotonicities arise is when the object is the right to compete in a research tournament analyzed by Fullerton and McAfee (1999). The private information is the cost of conducting research. The probability of winning the tournament is decreasing with own cost, but increasing with the opponents'. It turns out that the object value (the right to participate in the tournament) has the same feature. Thus, the standard assumption that the bidders' private informations are monotonic in the same direction is not fulfilled.

Non-monotonicities also arise as a consequence of multidimensional and complex information. For instance, in an oil-lease auction, the private information of bidders includes estimates of the track size, oil quality, extraction cost, future international prices, etc. These variables may compound in a non-monotonic fashion: small firms have larger costs to operate big tracts compared to big firms, but the reverse is true if the track size is small. Some of these variables are private values (such as the extraction cost), but others (such as the field size or the international oil price) are common values and the assessments of the opponents are likely to be important. Putting all these together, it is not surprising that non-monotonicities arise in such a multidimensional setting.[11]

We described cases where the value functions may be non-monotonic because of multidimensionality of (independent) types. On the other hand, an example presented by Reny and Zamir (2004) shows that multidimensionality and correlation of types may lead to nonmonotonic equilibria, even if the value functions are monotonic. In their example there are three bidders with bidimensional affiliated signals and monotonic value functions, but all equilibria are non-monotonic. de Castro (2007) suggests that general correlation of one-dimensional types may imply a small proportion of monotonic equilibria. Also, de Frutos and Pechlivanos (2006) consider an auction with independent bidimensional types, for which there exist no equilibrium in monotonic strategies (see their Proposition 2).

Multidimensionality is also a source of non-monotonicities through a different channel. Athey and Levin (2001) and Ewerhart and Fieseler (2003) consider single-object auctions where bids are multidimensional (although the information is unidimensional). Athey and Levin (2001) analyze the USA Forest Service timber auction where bids are the unit prices of each specie of timber. Before the auction the Forest Service (auctioneer) estimates the quantity of each specie in the tract and publicly announces them. Thus, the auctioneer receives the multidimensional bid or the "supply" curve

[11] In many cases, multidimensional information can be reduced to unidimensional types through a sufficient statistic, as suggested by Milgrom and Weber (1982). Nevertheless, this approach does not justify the monotonicity of the value function.

and obtains the bidders' scores by multiplying the submitted unit prices by the announced quantities. The highest score bidder is the winner. Nevertheless, the actual price paid by the winner is obtained by multiplying his unit price by the actual number of removed trees, which is verified *ex-post* by the Forest Service. Thus, a bidder who makes a better estimative of the number of trees of each specie and knows that the initial estimate of the Forest Service is not accurate, may strategically manipulate his offer. As a result, non-monotonic bids can arise.

A similar situation (see Ewerhart and Fieseler 2003) is a procurement auction for a service with equipment supply such that bidders should put prices on materials and work hours. Again, a scoring rule determines the winner through the seller's initial estimative (of materials and work hours). The final payment is made according to the actual use of materials and work hours. Again, strategic manipulation of bids leads to non-monotonic bidding functions.

All these examples show the importance of considering auctions with relaxed monotonicity assumptions. In the next section we analyze the multidimensional type space and bidding functions that may not be monotonic. For this setting we obtain results analogous to those found in Sect. 3 for monotonic unidimensional auctions. However, the characterization is not quite as complete as in the previous case, as we explain below.

5 Extension to multidimensional auctions

In this section we discuss how the results of Sect. 3 can be extended to N players and multidimensional type space S, which can be, for instance, a universal type space in the sense of Mertens and Zamir (1985). We consider strategies $b : S \to \mathbb{R}$ that induce ties with zero probability but are not necessarily monotonic even if S has a natural order, as in the case of $S \subset \mathbb{R}$. This will allow us to characterize when this kind of equilibria does not exist and ties are unavoidable.

Definition 1 *A bounded measurable function $b : S \to \mathbb{R}$ is regular if the c.d.f. $F_b(c) \equiv \Pr\{s \in S : b(s) \le c\}$ is absolutely continuous and strictly increasing in its support $\left[b_*, b^*\right]$. Let $\mathcal{S}$ denote the set of these regular functions.*

The set $\mathcal{S}$ is formed by functions b that do not induce ties with positive probability (because F_b is absolutely continuous) and that do not have gaps in the support of bids (because F_b is increasing). Observe that $\mathcal{S}$ contains non-monotonic bidding functions. As a first step, we will restrict attention to symmetric equilibria $(b, \ldots, b) \in \mathcal{S}^N$. For brevity and with some abuse of notation, we refer to such equilibria as $b \in \mathcal{S}$.

As in Sect. 3, our purpose is to characterize conditions on v such that a pure strategy equilibrium $b \in \mathcal{S}$ exists. We make some observations before the statement of the result.

If $b \in \mathcal{S}$ is an equilibrium, then the c.d.f. of the opponents' maximum bid, $\tilde{P} : \mathbb{R}_+ \to [0, 1]$ such that

$$\tilde{P}(c) = \Pr\left\{t_{-i} \in S^{N-1} : b\left(t_j\right) \le c, \forall j \ne i\right\}, \tag{9}$$

is increasing since b is regular. Define $\tilde{b} \equiv \tilde{P}^{-1}$ and $P = \tilde{P} \circ b$, so that $P : S \to [0, 1]$ is a reparametrization of the types in S satisfying

$$P(t_i) = \Pr\left\{t_{-i} \in S^{N-1} : b(t_j) \leqslant b(t_i), \forall j \neq i\right\}. \tag{10}$$

Therefore, P associates to each type her winning probability given $b \in \mathcal{S}$. As we show in the Appendix, this reparametrization has some interesting properties. In particular, $b = \tilde{b} \circ P$ is such that the (increasing) function $\tilde{b}$ is an equilibrium bidding function of a reduced auction with only two players, types $P(s) \in [0, 1]$ and utility given by

$$\tilde{v}(x, y) \equiv E[v(t_i, t_{-i}) | P(t_i) = x, P_{(-i)}(t_{-i}) = y], \tag{11}$$

where $P_{(-i)}(t_{-i}) \equiv \max_{j \neq i} P(t_j)$. Thus, we can consider instead the auction $\widetilde{\mathcal{A}} = \left(\tilde{S}, \tilde{\Sigma}, \tilde{\mu}, \tilde{N}, \tilde{v}\right)$, where $\tilde{S} = [0, 1]$, $\tilde{\Sigma}$ is the Borel σ-field, $\tilde{\mu}$ is the Lebesgue measure on $[0, 1]$, $\tilde{N} = 2$ and $\tilde{v}$ defined by (11) for a reparametrization P. The relation between auctions $\widetilde{\mathcal{A}}$ and $\mathcal{A}$ is explained by Proposition 10 in the Appendix.[12]

We have an analogous version of Theorem 1:

Theorem 4 *There exists a symmetric equilibrium in regular strategies for $\mathcal{A}$ if and only if there exists a reparametrization $P : S \to [0, 1]$ such that:*

(i) *there exists a strictly increasing and continuous function $\tilde{b} : [0, 1] \to \mathbb{R}$ satisfying the payment expression (1) for $\tilde{v}$ given by (11) using the reparametrization P;*
(ii) *for all $(x, y) \in [0, 1] \times [0, 1]$ and s such that $P(s) = x$,*

$$\int_y^x \left[\hat{v}(s, \alpha) - \tilde{v}(\alpha, \alpha)\right] d\alpha \geqslant 0, \tag{12}$$

where $\hat{v}(s, \alpha) \equiv E[v(s, t_{-i}) | P_{(-i)}(t_{-i}) = \alpha]$.

In the affirmative case, the function $\tilde{b}$ is an equilibrium of $\widetilde{\mathcal{A}}$.

The main difference between Theorem 1 and Theorem 4 is that the former does not need any reparametrization. In fact, Theorem 1 is a special case of Theorem 4 for $S = [0, 1]$ and the identity reparametrization. Another difference is that, in condition (ii) of Theorem 4 we use two different functions, $\hat{v}$ and $\tilde{v}$, whereas in Theorem 1 we used only v. To understand this difference, the reader should note that the auction

[12] The discussion that precedes Proposition 10 also clarifies this construction and shows that the reparametrization may be defined with no mention of b.

$\widetilde{\mathcal{A}}$ satisfies the assumptions of Theorem 1, replacing v by $\tilde{v}$. Thus, condition (ii) of Theorem 1 for $\widetilde{\mathcal{A}}$ is exactly the following:

$$\int_{y}^{x} \left[\tilde{v}\left(x,\alpha\right) - \tilde{v}\left(\alpha,\alpha\right)\right] d\alpha \geqslant 0. \tag{13}$$

In fact, this is sufficient for equilibrium existence for $\widetilde{\mathcal{A}}$ but is not sufficient for equilibrium existence for $\mathcal{A}$. To ensure equilibrium existence for $\mathcal{A}$, we need to ensure that each type s satisfying $P\left(s\right) = x$, should not have an incentive for deviation. Thus, we need to specify the above inequality for each of these s and not only for the reparametrized type x. This explains our definition of $\hat{v}$, which depends directly on s. It turns out that (12) implies (13), but in general the converse does not hold.

Theorem 4 implies that if there exists an equilibrium we have an outcome-equivalent auction $\widetilde{\mathcal{A}}$ with just two players and types uniformly distributed on [0, 1]. In other words, a multidimensional auction with an equilibrium can always be reduced to a simple unidimensional auction preserving some of the properties of the original one. This reinforces our interest in the auctions considered in Sect. 3: they are the outcome-equivalent auction of the multi-player and multidimensional auction considered in this section.

Note that Theorem 4 does not make trivial the task of verifying whether an equilibrium exists for a given auction $\mathcal{A}$, as Theorem 1 does for monotonic equilibria. Indeed, given a reparametrization $P : S \to [0,1]$, in general it is trivial to check whether conditions (i) and (ii) are satisfied. The non-trivial part is exactly finding which reparametrization works. Although we do not have a general method to find the reparametrization, the following proposition might be useful:

Proposition 3 *If $\mathcal{A}$ has a pure strategy equilibrium $b \in \mathcal{S}$ and there exists $\partial_b \Pi\left(\cdot,\cdot\right)$ at $\left(s, b\left(s\right)\right)$ for all s such that $b\left(s\right) \in \left(b_*, b^*\right)$, then the reparametrization map $P : S \to [0,1]$ must satisfy the following property: for all $s \in S$ such that $P\left(s\right) = x$, then*

$$\tilde{v}\left(x,x\right) = E[v(s,t_{-i})|P_{(-i)}(t_{-i}) = x], \tag{14}$$

where $\tilde{v}$ is defined by (11).[13]

This proposition can be used to compute explicitly the reparametrization, as the following example illustrates.

Example 4 Consider a symmetric first-price auction between two bidders with independent and uniformly distributed types in [0, 1]. The utility function is given by $v(x,y) = x + \alpha\left(x\right) y$, where $\alpha\left(x\right) = 3 - 4x + 2x^2$. Thus, $\partial_x v(x,y)$ can be negative.

[13] This condition is analogous to the one derived by Araujo and Moreira (2000) for screening problems and by Araujo and Moreira (2003) for signaling games. In these papers, the violation of the single crossing property leads to non-monotonicity.

Although

$$b(x) = \frac{1}{x}\int_0^x v(\alpha,\alpha)\,d\alpha = \frac{x\left(24-16x+3x^2\right)}{12}$$

is increasing in [0, 1], condition (8)

$$\int_y^x [v(x,\alpha) - v(\alpha,\alpha)]\,d\alpha = \frac{(x-y)^2}{6}\left[3+3x^2-8y+3y^2+x(-4+6y)\right] \geqslant 0$$

does not hold (for instance, take $x = 0$ and $y = 1$). Theorem 1 implies that there is no monotonic equilibrium for $\mathcal{A}$. Nevertheless, in the Appendix we illustrate how to use Proposition 3 to construct a U-shaped symmetric equilibrium where ties occur with zero probability.

Completely ordered auctions

Now we present the analogous version of Theorem 2 for this setting. First, we need a generalization of Assumption 4:

Assumption 5 $v(t_i, t_{-i})$ is such that if $v(t_i, t_{-i}) < v\left(t_i', t_{-i}\right)$ for some t_{-i}, then $v\left(t_i, t_{-i}'\right) < v\left(t_i', t_{-i}'\right)$ for all t_{-i}'.

When $S = [0, 1]$, Assumption 5 is strictly weaker than Assumption 4, which is already a generalization of the standard assumptions of the auction theory. However, this does not mean that Assumption 5 is weak in multidimensional settings and, indeed, it can be quite restrictive. To see this, consider the function $\bar{v}(s) \equiv E_{t_{-i}}\left[v(s, t_{-i})\right]$ and the natural complete order in S induced by it: $s' \succcurlyeq s \Leftrightarrow \bar{v}\left(s'\right) \geq \bar{v}(s)$. Under Assumption 5, this order is equivalent to $s' \succcurlyeq_{t_{-i}} s \Leftrightarrow v\left(s', t_{-i}\right) \geq v(s, t_{-i})$, for any t_{-i}. Thus, there is a unique way to define the reparametrization under Assumption 5:

$$P(t_i) \equiv \Pr\left\{t_{-i} \in S^{N-1} : \bar{v}\left(t_j\right) < \bar{v}(t_i),\, j \neq i\right\}. \tag{15}$$

i.e., it is the winning probability with respect to that order.

However, as in the case of Assumption 4, Assumption 5 does not imply equilibrium existence. Theorem 5 below (a generalization of Theorem 2) shows that $\tilde{b}$ satisfying the payment expression (1) and being an increasing function is sufficient, under Assumption 5.

Theorem 5 (Necessary and Sufficient Condition for Equilibrium Existence) *Consider an auction $\mathcal{A}$ satisfying Assumptions 1, 2, 3 and 5. Let P be defined by (15) and $\tilde{v}$ by (11) for this P. If $\tilde{v}$ is continuous, then there exists an equilibrium for $\mathcal{A}$ if and only if there exists an increasing function $\tilde{b}$ that satisfies (1) with $\tilde{v}$ in the place of v. In the affirmative case, the equilibrium of $\mathcal{A}$ is given by $b = \tilde{b} \circ P$. If there is a unique*

$\tilde{b}$ that satisfies such a property, the equilibrium of $\mathcal{A}$ is also unique in $\mathcal{S}$. If $\tilde{v}$ is not continuous, the condition is still sufficient.

As commented after Theorem 4 , multidimensional auctions with regular equilibria can always be reduced (through reparametrizations) to unidimensional auctions. Thus, to prove equilibrium existence for multidimensional auctions, it suffices to "reduce" them to unidimensional auctions. This reduction is not an easy task in general, but it is trivial under Assumption 5 because types can be ordered in a unique and natural way.[14]

Under Assumption 5, the all-pay auction tie-breaking rule also works. We then have a simple generalization of Theorem 3.

Theorem 6 (Equilibrium Existence with Ties) *Consider an auction $\mathcal{A}$ where the all-pay auction tie-breaking rule is adopted. If there is an absolutely continuous function $\tilde{b}$ that satisfies* (1)*, then there exists a pure strategy equilibrium.*

6 Conclusion

Simon and Zame (1990) and JSSZ proposed a solution to the problem of equilibrium existence for discontinuous games when the standard tie-breaking rule is not sufficient to ensure it. Araujo and de Castro (2007) follow their idea and prove equilibrium existence for single and double asymmetric auctions with interdependent values.

Our paper characterizes when special tie-breaking rules are really needed. We accomplish this task for the set of symmetric auctions with unidimensional types and partially for a set of multidimensional symmetric auctions with weak monotonicity assumption.

When a special tie-breaking rule is needed, we show that the all-pay auction tie-breaking rule is sufficient to ensure equilibrium existence. This suggests that the all-pay auction (and the war of attrition) are better mechanisms to reveal information than first- and second-price auctions.

Our results suggest the possibility of completely defining the payoff of a discontinuous game to ensure equilibrium existence, even when the standard definition does not ensure this. In sum, this paper is a first step in the direction of finding general and exogenously defined tie-breaking rules that do not require communication of private information and guarantee equilibrium existence.

Acknowledgments We are grateful to the seminar participants at the NBER/NSF Decentralization Conference 2005, the 2005 Stony Brook Game Theory Festival, the 2004 Latin American Meeting, the 2004 European Meeting of the Econometric Society, University of Toulouse 1, IMPA, Getulio Vargas Foundation and especially to Michel Le Breton, David McAdams, Flavio Menezes, Paulo K. Monteiro and Sergio O. Parreiras. We thank CNPq of Brazil for the support of this research.

[14] The reduction-type dimension is not a novelty in auction theory. Studying efficient auctions, Dasgupta and Maskin (2000) use a close condition to Assumption 5 and Jehiel et al. (1996) make such a reduction for revenue maximization.

Appendix

For convenience we follow the notation of Sect. 5, but first we analyze auctions between two players with independent and uniformly distributed types in [0, 1] and satisfying Assumptions 1 and 2. We allow for the existence of reserve prices that exclude bidders with types in $[0, x_0)$ from the auction, that is, we relax Assumption 3 to Assumption 3' below.

Let $\tilde{\mathcal{S}}$ be the set of non-decreasing functions $\tilde{b} : [0,1] \to \{b_{OUT}\} \cup [b_{\min}, +\infty)$ such that there exists $x_0 \in [0,1]$ satisfying: $\tilde{b}([0, x_0)) = \{b_{OUT}\}$ and $\tilde{b}$ is strictly increasing in $(x_0, 1]$. For $\tilde{b} \in \tilde{\mathcal{S}}$, let $b_* = \inf\{\tilde{b}(x) \geqslant b_{\min} : x \in [0,1]\}$ and $b^* = \sup\{\tilde{b}(x) \geqslant b_{\min} : x \in [0,1]\}$. Note that $x_0 = 0$ when there is no exclusion.

The interim payoff of a bidder with type x who bids $\beta \geqslant b_{\min}$ and faces an opponent following $\tilde{b} \in \tilde{\mathcal{S}}$ is

$$\tilde{\Pi}\left(x, \beta, \tilde{b}\right) = \int_0^{\tilde{b}^{-1}(\beta)} \left[\tilde{v}(x, \alpha) - p^W\left(\beta, \tilde{b}(\alpha)\right)\right] d\alpha - \int_{\tilde{b}^{-1}(\beta)}^1 p^L\left(\beta, \tilde{b}(\alpha)\right) d\alpha, \tag{16}$$

where $\tilde{b}^{-1}(\beta) = \inf\left\{x \in [0,1] : \tilde{b}(x) \geqslant \beta\right\}$. Let us also define

$$\tilde{p}\left(\beta, \tilde{b}\right) = \int_0^{\tilde{b}^{-1}(\beta)} p^W\left(\beta, \tilde{b}(\alpha)\right) d\alpha + \int_{\tilde{b}^{-1}(\beta)}^1 p^L\left(\beta, \tilde{b}(\alpha)\right) d\alpha.$$

Assumption 2 requires that $p^L(b_{\min}, b) = p^L\left(b_{\min}, b'\right)$ for all b and b' and $p^W(\cdot, b_{OUT}) = p^W(\cdot, b_{\min})$. Thus, $\alpha < x$ implies $\tilde{b}(\alpha) \leqslant b_{\min}$ and the expression of $\tilde{p}\left(b_{\min}, \tilde{b}\right)$ can be simplified to

$$\int_0^{\tilde{b}^{-1}(b_{\min})} p^W(b_{\min}, b_{\min})\, d\alpha + \int_{\tilde{b}^{-1}(b_{\min})}^1 p^L(b_{\min}, b_{\min})\, d\alpha.$$

We assume:

Assumption 3' There exists $x_0 \in [0,1]$ such that, alternatively,

(i) $x_0 = 0$ and

$$\int_0^x \tilde{v}(x, \alpha)\, d\alpha \geqslant \int_0^x p^W(b_{\min}, b_{\min})\, d\alpha + \int_x^1 p^L(b_{\min}, b_{\min})\, d\alpha$$

for all $x \in [0,1]$ or

(ii) $x_0 > 0$,

$$\int_0^{x_0} \tilde{v}(x_0, \alpha)\, d\alpha = \int_0^{x_0} p^W(b_{\min}, b_{\min})\, d\alpha + \int_{x_0}^{1} p^L(b_{\min}, b_{\min})\, d\alpha$$

and $x < x_0 < y$ implies

$$\int_0^{x} \tilde{v}(x, \alpha)\, d\alpha \leqslant \int_0^{x_0} \tilde{v}(x_0, \alpha)\, d\alpha \leqslant \int_0^{y} \tilde{v}(y, \alpha)\, d\alpha.$$

Note that Assumption 3'(i) corresponds to the original Assumption 3. In Assumption 3'(ii), x_0 is the type who bids the minimum bid $b_{\min}$. This type necessarily has zero expected payoff, because she must be indifferent to participating or not. The inequalities in Assumption 3'(ii) are weak monotonicity conditions that imply that types below x_0 do not have the incentive to bid above $b_{\min}$ since the object value is not greater than expected payments.

Proposition 4 *Suppose that Assumption 2(i) holds. Let $\tilde{b} \in \tilde{S}$ be an equilibrium increasing in $(x_0, 1)$. Then, $\tilde{b}$ is continuous on $(x_0, 1)$. Moreover, if $\tilde{v}$ is continuous in the second variable, then $\tilde{b}$ is differentiable in $(x_0, 1)$ and satisfies*

$$\tilde{b}'(x) = \frac{\tilde{v}(x, x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right)}{E_\alpha\left[\partial_1 p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) > \tilde{b}(\alpha)\right]} + \partial_1 p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) < \tilde{b}(\alpha)\right]}\right]}. \tag{17}$$

Proof The proof is an adaptation of the proof of Theorem 2 in Maskin and Riley (1984). It is available upon request to the authors. □

Proposition 5 *Assume that $\tilde{v}$ is continuous and Assumption 2(ii) holds. Let $\tilde{b} \in \tilde{S}$ be an increasing equilibrium in $(x_0, 1)$. Then, $\tilde{b}$ is continuous in $(x_0, 1)$ and*

$$\tilde{v}(x, x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right) = 0, \forall x \in (x_0, 1). \tag{18}$$

Proof The proof is based on the proof of Theorem 3 of Maskin and Riley (1984). It is also available upon request to the authors. □

We have the following:

Lemma 6 *Assume that $\tilde{v}$ is continuous. Let $\tilde{b} \in \tilde{S}$ be an increasing equilibrium in $(x_0, 1)$. Then, $\tilde{\Pi}\left(x, \cdot, \tilde{b}\right)$ is differentiable in (b_*, b^*).*

Proof Under Assumption 2(i), Proposition 4 implies that $\tilde{b}$ is differentiable and, since $\tilde{b}$ is increasing on $(x_0, 1)$, $\tilde{b}^{-1}$ is differentiable on (b_*, b^*). We also have that $\tilde{v}$ is continuous and p^W and p^L are differentiable. Thus, one can easily see from (16) that

$$\begin{aligned}\partial_\beta \tilde{\Pi}\left(x, \beta, \tilde{b}\right) &= \tilde{v}\left(x, \tilde{b}^{-1}(\beta)\right) - p^W(\beta, \beta) - p^L(\beta, \beta) \\ &\quad - \int_0^{\tilde{b}^{-1}(\beta)} \partial_1 p^W\left(\beta, \tilde{b}(\alpha)\right) \cdot \left[\tilde{b}^{-1}(\beta)\right]' d\alpha \\ &\quad - \int_{\tilde{b}^{-1}(\beta)}^{1} \partial_1 p^L\left(\beta, \tilde{b}(\alpha)\right) \cdot \left[\tilde{b}^{-1}(\beta)\right]' d\alpha.\end{aligned}$$

Under Assumption 2(ii), $\partial_1 p^W(\cdot,\cdot) = \partial_1 p^L(\cdot,\cdot) = 0$ and $\partial_\beta \tilde{\Pi}\left(x, \beta, \tilde{b}\right)$ exists, with

$$\partial_\beta \tilde{\Pi}\left(x, \beta, \tilde{b}\right) = \tilde{v}\left(x, \tilde{b}^{-1}(\beta)\right) - p^W(\beta, \beta) - p^L(\beta, \beta).$$

. □

Now let us consider cases where $\tilde{b}$ is not monotonic since this is exactly the setting of Theorem 3. To treat non-increasing $\tilde{b}$, we define the following:

Modified auction: Fix $\tilde{b}$ such that $\tilde{b}(x_0) = b_{\min}$ and $\tilde{b}(y) > b_{\min}$ for $y > x_0$. The bidder bids a type $y \in [0, 1]$. If $y < x_0$, the payment is zero. If $y \geq x_0$, the payment is determined as if the bidder has submitted the bid $\tilde{b}(y)$. The bidder wins against opponents who announce types below y and loses against those who announce types above y. If there is a tie, the object is given to each bidder with probability $1/2$.

Observe that if $\tilde{b}$ is increasing, the modified auction is the standard auction where all bidders follow $\tilde{b}$. If $\tilde{b}$ is not increasing, the difference is that the winning events are not determined by $\tilde{b}$ but by the announced type y. The rule of the modified auction implies the following interim payoff:

$$\hat{\Pi}(x, y) = \begin{cases} \int_0^y \left[\tilde{v}(x, \alpha) - p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right)\right] d\alpha & \\ \quad - \int_y^1 p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha, & \text{if } y \geqslant x_0 \\ 0 & \text{if } y < x_0. \end{cases}$$

We can simplify the above expression to

$$\hat{\Pi}(x, y) = \begin{cases} \int_0^y \tilde{v}(x, \alpha)\, d\alpha - \hat{p}(y), & \text{if } y \geqslant x_0 \\ 0 & \text{if } y < x_0 \end{cases} \tag{19}$$

where

$$\hat{p}(y) \equiv \begin{cases} \int_0^y p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha + \int_y^1 p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha, & \text{if } y \geqslant x_0 \\ 0, & \text{if } y < x_0. \end{cases}$$

Proposition 7 *(Payment Rule) Assume that $\tilde{v}$ is continuous. If truth-telling is an equilibrium for the modified auction, then*

$$\hat{p}(y) = \begin{cases} \hat{p}(x_0) + \int_{x_0}^{y} \tilde{v}(\alpha, \alpha)\, d\alpha, & \text{if } y > x_0 \\ \int_0^{x_0} \tilde{v}(x_0, \alpha)\, d\alpha, & \text{if } y = x_0 \\ 0, & \text{if } y < x_0. \end{cases} \tag{20}$$

Proof In the case of Assumption 2(i), $\tilde{b}$, p^W and p^L are differentiable on $(x_0, 1)$. Thus, $\hat{p}$ and $\hat{\Pi}$ are also differentiable. So, for every $y \in (x_0, 1)$, we have

$$\hat{p}'(y) = \partial_y \left\{ \int_0^y \tilde{v}(x, \alpha)\, d\alpha - \hat{\Pi}(x, y) \right\} = \tilde{v}(x, y) - \partial_y \hat{\Pi}(x, y).$$

The first-order condition for truth-telling optimality gives for $x \geqslant x_0$, $\partial_2 \hat{\Pi}(x, y)\,|_{y=x} = 0$, which implies that $\hat{p}'(x) = \tilde{v}(x, x)$. Thus,

$$\hat{p}(y) = \hat{p}(x_0) + \int_{x_0}^{y} \tilde{v}(\alpha, \alpha)\, d\alpha$$

for $y \geqslant x_0$. Now, let us turn to Assumption 2(ii). Since $\tilde{b}$ is only continuous, $\hat{p}$ is not necessarily differentiable. Nevertheless, if $y \geqslant x_0$,

$$\begin{aligned} \hat{p}(y) &= \int_0^y p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha + \int_y^1 p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha \\ &= \int_{x_0}^{y} \left[p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right) - p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right)\right] d\alpha + \hat{p}(x_0) \\ &= \int_{x_0}^{y} h\left(\tilde{b}(\alpha)\right) d\alpha + \hat{p}(x_0) \\ &= \int_{x_0}^{y} \tilde{v}(\alpha, \alpha)\, d\alpha + \hat{p}(x_0), \end{aligned}$$

where $h(z) \equiv p^W(b, z) - p^L\left(b', z\right)$ does not depend on b and b' by Assumption 2(ii). For $y < x_0$, the payment is zero by the definition of the modified auction. For $y = x_0$, $\hat{p}(y)$ is obtained from Assumption 3'. □

Now we turn to the equilibrium existence. We will not assume that $\tilde{v}$ is continuous. Instead, we assume only the validity of the payment expression (20).

Proposition 8 *(Equilibrium) Assume* (20)*. Then, truth-telling is an equilibrium of the modified auction if and only if*

$$\begin{cases} \int_y^x \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha \geqslant 0, & \text{if } x, y \geqslant x_0 \\ \int_{x_0}^x \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha + \int_0^{x_0} \left[\tilde{v}(x,\alpha) - \tilde{v}(x_0,\alpha)\right] d\alpha \geqslant 0, & \text{if } x \geqslant x_0 > y \\ 0 \geqslant \int_{x_0}^y \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha + \int_0^{x_0} \left[\tilde{v}(x,\alpha) - \tilde{v}(x_0,\alpha)\right] d\alpha & \text{if } y \geqslant x_0 > x. \end{cases} \tag{21}$$

Proof Given (20), the optimity condition for truth-telling, namely $\hat{\Pi}(x,x) - \hat{\Pi}(x,y) \geqslant 0$, is equivalent to

$$\int_0^x \tilde{v}(x,\alpha)\,d\alpha - \int_{x_0}^x \tilde{v}(\alpha,\alpha)\,d\alpha - \hat{p}(x_0) - \int_0^y \tilde{v}(x,\alpha)\,d\alpha + \int_{x_0}^y \tilde{v}(\alpha,\alpha)\,d\alpha$$

$$+\hat{p}(x_0) = \int_y^x \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha \geqslant 0$$

if $x, y \geqslant x_0$. The other cases are similar. □

As previously stated, if $\tilde{b}$ is increasing, the modified auction is just the original (unmodified) auction. Thus, we have:

Corollary 9 *Assume that $\tilde{b}$ is increasing on $(x_0, 1)$ and implies* (20)*. Then, $\tilde{b}$ is an equilibrium of the auction $\widetilde{\mathcal{A}}$ if and only if* (21) *holds.*

Observe that Proposition 8 and Corollary 9 do not require $\tilde{v}$ to be continuous.

Definition of the reparametrized auction

Fix $b \in \mathcal{S}$. Define the map $P^b = P$ by (10), that is,

$$P(t_i) = \Pr\left\{t_{-i} \in S^{N-1} : b\left(t_j\right) \leqslant b(t_i), \forall j \neq i\right\}.$$

P associates each type to the winning probability given $b \in \mathcal{S}$. From the symmetry, P does not depend on i and $P(t_i) \lesseqqgtr P\left(t_j\right)$ if and only if $b(t_i) \lesseqqgtr b\left(t_j\right)$. Obviously, two players have the same probability of winning if and only if they choose the same bid. Thus,

$$\left\{t_{-i} \in S^{N-1} : b_{(-i)}(t_{-i}) < b(t_i)\right\} = \left\{t_{-i} \in S^{N-1} : P_{(-i)}(t_{-i}) < P(t_i)\right\},$$

where $P_{(-i)}(t_{-i}) \equiv \max_{j \neq i} P^b\left(t_j\right)$. The equality of these events and (10) imply that

$$P(t_i) = \Pr\left\{t_{-i} \in S^{N-1} : P_{(-i)}(t_{-i}) < P(t_i)\right\}.$$

This can be used as a definition of an admissible reparametrization, even if the bidding function b is not given. This and the fact that the range of P is [0, 1] imply

$$\Pr\left\{t_{-i} \in S^{N-1} : P_{(-i)}(t_{-i}) < c\right\} = c, \tag{22}$$

for all $c \in [0, 1]$. The above equation means that the distribution of $P_{(-i)}(t_{-i})$ is uniform on [0, 1].

The c.d.f. of the opponents' maximum bid, $\tilde{P}$, is given by (9) and $\tilde{b} = \tilde{P}^{-1}$. Since $P(t_i) = \tilde{P}(b(t_i))$, $b(t_i) = \tilde{P}^{-1} \circ P(t_i) = \tilde{b} \circ P(t_i)$. Moreover, $\tilde{b}$ strictly increasing implies that

$$E\left[v(t_i, \cdot) | t_i = s, b_{(-i)}(t_{-i}) = \beta\right] = E\left[v(t_i, \cdot) | t_i = s, P_{(-i)}(t_{-i}) = \beta\right]. \tag{23}$$

If $b : S \to \mathbb{R}$ is an equilibrium of $\mathcal{A}$, $\Pi(t_i, b(t_i)) \geq \Pi(t_i, c)$, for all $c \in \mathbb{R}$, where $\Pi(t_i, c)$ is the interim payoff of $\mathcal{A}$. Let us define

$$\tilde{\Pi}(x, c) \equiv E\left[\Pi(t_i, c) | P(t_i) = x\right]. \tag{24}$$

The notation suggests that $\tilde{\Pi}(x, c)$ is the interim payoff of $\widetilde{\mathcal{A}}$. This is exactly the content of Proposition 10 below which justifies the interpretation of $\widetilde{\mathcal{A}}$ as the outcome-equivalent of $\mathcal{A}$, as mentioned in Sect. 5. Since we can prove the equality of the *interim* payoffs, there is no loss of generality to work with uniform distribution of (reparametrized) types on [0, 1].

Proposition 10 *Consider an auction $\mathcal{A}$ satisfying Assumptions 1, 2 and 3 and $\widetilde{\mathcal{A}}$ defined above for a given $P : S \to [0, 1]$. Following the previous notation, we have:*

(i)

$$\tilde{\Pi}(x, c) = \int_0^{\tilde{b}^{-1}(c)} \left[\tilde{v}(x, \alpha) - p^W\left(c, \tilde{b}(\alpha)\right)\right] d\alpha - \int_{\tilde{b}^{-1}(c)}^{1} p^L\left(c, \tilde{b}(\alpha)\right) d\alpha. \tag{25}$$

(ii) *Suppose that $\tilde{b}$ is strictly increasing and satisfies the payment expression (20). If, for all $(x, y) \in [0, 1] \times [0, 1]$ and all s such that $P(s) = x$,*

$$\int_y^x \left[\hat{v}(s, \alpha) - \tilde{v}(\alpha, \alpha)\right] d\alpha \geqslant 0, \tag{26}$$

where $\hat{v}(s, \alpha) \equiv E[v(s, t_{-i}) | P_{(-i)}(t_{-i}) = y]$, then $\tilde{b}$ is an equilibrium of $\widetilde{\mathcal{A}}$ and $b = \tilde{b} \circ P$ is an equilibrium of $\mathcal{A}$.

Proof Let us introduce the following notation:

$$\Pi^{+}(t_i, c) = \int \left[v(t_i, \cdot) - p^W\left(c, b_{(-i)}(\cdot)\right)\right] 1_{[c > b_{(-i)}(\cdot)]} \Pi_{j \neq i} \mu\left(dt_j\right),$$

$$\Pi^{-}(t_i, c) = \int p^L\left(c, b_{(-i)}(\cdot)\right) 1_{[c < b_{(-i)}(\cdot)]} \Pi_{j \neq i} \mu\left(dt_j\right),$$

$$\tilde{\Pi}^{+,-}(\phi, c) \equiv E\left[\Pi_i^{+,-}(t_i, c) \,|\, P(t_i) = \phi\right].$$

Let us start with the proof for $\tilde{\Pi}_i^{+}$ and Π_i^{+}. Denote the conditional expectation by

$$g^{t_i, c}(\alpha) \equiv E\left[v(t_i, t_{-i}) - p^W\left(c, b_{(-i)}(t_{-i})\right) | P^b_{(-i)}(t_{-i}) = \alpha\right]. \tag{27}$$

The event $\left[c > b_{(-i)}(t_{-i})\right]$ occurs if and only if $\left[\tilde{P}^b(c) > P^b_{(-i)}(t_{-i})\right]$ occurs. Then,

$$\Pi^{+}(t_i, c) = \int g^{t_i, c}\left(P^b_{(-i)}(t_{-i})\right) 1_{\left[\tilde{P}^b(c) > P^b_{(-i)}(t_{-i})\right]} \Pi_{j \neq i} \mu\left(dt_j\right).$$

Now we appeal to Lehmann (1959), Lemma 2.2, p. 43. It says that if R is a transformation and $\mu^*(B) = \mu\left(R^{-1}(B)\right)$, then

$$\int_{R^{-1}(B)} g[R(t)] \mu(dt) = \int_B g(\alpha) \mu^*(d\alpha).$$

In our case, $R = P^b_{(-i)}$ and $\mu^*([0, c]) = \mu^*([0, c)) = \tau_{-i}\left(\left(P^b\right)^{-1}_{(-i)}([0, c))\right) = \Pr\{t_{-i} \in S^{N-1} : P^b\left(t_j\right) < c, \forall j \neq i\} = c$, by (22). So, μ^* is exactly the Lebesgue measure, and

$$\Pi^{+}(t_i, c) = \int_0^{\tilde{P}^b(c)} g^{t_i, c}(\alpha) \, d\alpha. \tag{28}$$

From this and the definition of $\tilde{\Pi}^{+}$, we have

$$\begin{aligned}\tilde{\Pi}^{+}(\phi, c) &= E\left[\int_{0}^{\tilde{P}^{b}(c)} g^{t_i,c}(\alpha)\, d\alpha | P^{b}(t_i) = \phi\right] \\ &= \int_{0}^{\tilde{P}^{b}(c)} E\left[g^{t_i,c}(\alpha)\,|P^{b}(t_i) = \phi\right] d\alpha \\ &= \int_{0}^{\tilde{P}^{b}(c)} \left[\tilde{v}(\phi, \alpha) - p^{W}\left(c, \tilde{b}(\alpha)\right)\right] d\alpha,\end{aligned}$$

where the second line comes from Fubbini's Theorem and the last results from independency and the definition of $\tilde{v}(\phi, \alpha)$ and $g^{t_i,c}(\alpha)$ [see (11) and (27)]. From the fact that $\tilde{b} = \left(\tilde{P}^{b}\right)^{-1}$, we can substitute $\tilde{P}^{b}$ to obtain

$$\tilde{\Pi}^{+}(\phi, c) = \int_{0}^{\tilde{b}^{-1}(c)} \left[\tilde{v}(\phi, \alpha) - p^{W}\left(c, \tilde{b}(\alpha)\right)\right] d\alpha. \tag{29}$$

Now, we can repeat the same procedures for $\Pi^{-}(\phi, c)$ and obtain

$$\tilde{\Pi}^{-}(\phi, c) = \int_{\tilde{b}^{-1}(c)}^{1} p^{L}\left(c, \tilde{b}(\alpha)\right) d\alpha. \tag{30}$$

Adding up, we obtain the interim payoff of the indirect auction $\tilde{\Pi}(\phi, c) = \tilde{\Pi}^{+}(\phi, c) - \tilde{\Pi}^{-}(\phi, c)$.

For the second part, taking conditional expectations (26) implies that

$$\int_{y}^{x} \left[\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)\right] d\alpha \geqslant 0.$$

Thus, if $\tilde{b}$ is a strictly increasing function satisfying the payment expression (20), Corollary 9 implies that $\tilde{b}$ is an equilibrium of $\widetilde{\mathcal{A}}$. Now, we prove that $\Pi(s, b(s)) \geq \Pi(s, c)$, for all c, where $b = \tilde{b} \circ P$. This inequality is equivalent to

$$\int_{\{s_{-i} \in S: b(s_j) < b(s), \forall j\}} v(s, s_{-i})\, \mu(ds_{-i}) - p(b(s)) \geqslant \int_{\{s_{-i} \in S: b(s_j) < c, \forall j\}} \times v(s, s_{-i})\, \mu(ds_{-i}) - p(c),$$

or, if y is such that $\tilde{b}(y) = c$,

$$\int\limits_{\{z\in[0,1]:z<P(s)\}} \hat{v}(s,z)\,dz - p\left(\tilde{b}(P(s))\right) \geqslant \int\limits_{\{z\in[0,1]:z<y\}} \times \hat{v}(s,s_{-i})\,\mu(ds_{-i}) - p\left(\tilde{b}(y)\right).$$

Since $\tilde{b}$ satisfies (20), when $P(s) = x$ the above inequality can be written as

$$\int\limits_{y}^{x} \left[\hat{v}(s,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha \geqslant 0,$$

which is condition (26). □

Proof of Proposition 3. We need the following:

Lemma 11 (Payoff Characterization) *Consider an auction $\mathcal{A}$ satisfying Assumptions 1, 2 and 3. Fix $b \in \mathcal{S}$. The bidder i's payoff can be expressed by*

$$\Pi(t_i, b_i, b(\cdot)) = \Pi_i(t_i, b_*) + \int\limits_{(b_*, b_i)} \partial_{b_i}\Pi(t_i, \beta, b(\cdot))d\beta,$$

where $\partial_{b_i}\Pi(t_i, \beta, b(\cdot))$ exists for almost all β with

$$\begin{aligned}\partial_{b_i}\Pi(t_i,\beta,b(\cdot)) = E\Big[&-\partial_1 p^W\left(\beta, b_{(-i)}(t_{-i})\right) 1_{[\beta > b_{(-i)}(t_{-i})]} \\ &-\partial_1 p^L\left(\beta, b_{(-i)}(t_{-i})\right) 1_{[\beta < b_{(-i)}(t_{-i})]}\Big] + E[v(t_i,t_{-i}) \\ &-p^W(\beta,\beta) + p^L(\beta,\beta)\,|b_{(-i)}(t_{-i}) = \beta] f_{b_{(-i)}}(\beta),\ a.e.\end{aligned}$$

If $b \in \mathcal{S}$, it defines a reparametrization P^b by (10). The bid $b(t_i) = \beta$ is optimal for bidder t_i against the strategy $b(\cdot)$ of the opponents. Thus, $\partial_b\Pi(s, b(s)) = 0$ and Lemma 11 imply that

$$E\left[v(t_i,\cdot)\,|t_i = s, b_{(-i)}(t_{-i}) = \beta\right] = p^W(\beta,\beta) - p^L(\beta,\beta) - \frac{E_{t_{-i}}\left[\partial_{b_i} p^W 1_{[b_i > \mathbf{b}_{(-i)}]} + \partial_{b_i} p^L 1_{[b_i < \mathbf{b}_{(-i)}]}\right]}{f_{b_{(-i)}}(\beta)}.$$

Observe that the right-hand side does not depend on s (it depends on it only through the optimum bid $\beta = b(s)$). Thus, the left-hand side has to be the same for all s that make the same bid in equilibrium, which implies (14).

Through the proof of Theorem 5, we will make successive use of the following:

Lemma 12 *For any σ-field Ξ on S^{N-1}, we have*

$$\begin{aligned}\exists t_{-i} : v\left(s', t_{-i}\right) > v\left(s, t_{-i}\right) \Leftrightarrow \\ \forall t_{-i} : v\left(s', t_{-i}\right) > v\left(s, t_{-i}\right) \Leftrightarrow E\left[v\left(t_i, t_{-i}\right) | t_i = s', \Xi\right] \\ > E\left[v\left(t_i, t_{-i}\right) | t_i = s, \Xi\right], \ a.e.\end{aligned}$$

Proof Assumption 5 gives the first equivalence. By Assumption 3, if $v\left(s', t_{-i}\right) > v\left(s, t_{-i}\right)$, $\forall t_{-i}$, then, for any Ξ , $E\left[v\left(s', t_{-i}\right) - v\left(s, t_{-i}\right) | \Xi\right] > 0$ almost surely.[15] On the other hand, $E[v\left(t_i, t_{-i}\right) \mid t_i = s', \Xi] > E[v\left(t_i, t_{-i}\right) \mid t_i = s, \Xi]$ a.e. implies that $\exists t_{-i}$ such that $v\left(s', t_{-i}\right) > v\left(s, t_{-i}\right)$. □

Proof of Theorem 5 Equilibrium Existence Let P be defined by (15), $x, y \in [0, 1]$ and $s \in S$ be such that $P(s) = x$. We claim that Assumption 5 implies

$$\int_y^x \left[\hat{v}\left(s, \alpha\right) - \tilde{v}\left(\alpha, \alpha\right)\right] d\alpha \geqslant 0.$$

Indeed, if $x > y$, for all t_i and t_i' such that $P\left(t_i'\right) = x$ and $P\left(t_i\right) = y$, we have $v\left(t_i', t_{-i}\right) > v\left(t_i, t_{-i}\right)$ for all t_{-i}, by Assumption 5. Then, for all $z \in [0, 1]$,

$$\begin{aligned}\hat{v}\left(s, z\right) &\equiv E\left[v\left(t_i, t_{-i}\right) | P_{(-i)}\left(t_{-i}\right) = z\right] \\ &> E\left[v\left(t_i, t_{-i}\right) | P\left(t_i\right) = y, P_{(-i)}\left(t_{-i}\right) = z\right] = \tilde{v}\left(y, z\right).\end{aligned}$$

Then, if $y < \alpha < x = P(s)$, $\hat{v}\left(s, \alpha\right) - \tilde{v}\left(\alpha, \alpha\right) > 0$ and

$$\int_y^x \left[\hat{v}\left(s, \alpha\right) - \tilde{v}\left(\alpha, \alpha\right)\right] d\alpha \geqslant 0.$$

If $P(s) = x < \alpha < y$, we have $\hat{v}\left(s, \alpha\right) - \tilde{v}\left(\alpha, \alpha\right) < 0$ so that the same inequality is satisfied and the claim is proved.

By Theorem 4, $\tilde{b}$ is an equilibrium of $\widetilde{\mathcal{A}}$ and $b = \tilde{b} \circ P$ is an equilibrium of $\mathcal{A}$.

Now assume that $\tilde{v}$ is continuous. The sufficiency was already established. We now prove the necessity.

Let $b \in \mathcal{S}$ be an equilibrium of $\mathcal{A}$, P^b [given by (10)] be its associate reparametrization and $\tilde{b} = b \circ \left(P^b\right)^{-1}$. By Proposition 10, $\tilde{b}$ is an equilibrium of $\widetilde{\mathcal{A}}$.

Define $V(x) = E\left[v\left(t_i, t_{-i}\right) | P^b\left(t_i\right) = x\right]$. Then, we have:

Lemma 13 $x > y \Rightarrow V(x) \geq V(y)$.

Proof By absurd, suppose that there exist x and y, $x > y$, such that $V(x) < V(y)$.

First, we claim that for all t_i and t_i' such that $P^b\left(t_i\right) = x$ and $P^b\left(t_i'\right) = y$, we have $v\left(t_i, t_{-i}\right) < v\left(t_i', t_{-i}\right)$ for all t_{-i}. Otherwise, $v\left(t_i, t_{-i}\right) \geqslant v\left(t_i', t_{-i}\right)$ for some t_{-i} and,

15 See, for instance, Kallenberg (2002), Theorem 6.1, p. 104.

by Assumption 5, $v\left(t_i, t'_{-i}\right) \geqslant v\left(t'_i, t'_{-i}\right)$ for all t'_{-i}. Then, Lemma 12 would imply that $V(x) = E\left[v(t_i, t_{-i}) \,|\, P^b(t_i) = x\right] \geqslant E\left[v(t_i, t_{-i}) \,|\, P^b(t_i) = y\right] = V(y)$, a contradiction. Thus, the claim is proved.

This claim and Lemma 12 imply that

$$\tilde{v}(x, z) \equiv E\left[v(t_i, t_{-i}) \,|\, P^b(t_i) = x,\, P^b_{(-i)}(t_{-i}) = z\right]$$
$$< E\left[v(t_i, t_{-i}) \,|\, P^b(t_i) = y,\, P^b_{(-i)}(t_{-i}) = z\right] = \tilde{v}(y, z),$$

for all $z \in [0, 1]$, a.e. Thus,

$$\int_y^x \left[\tilde{v}(x, \alpha) - \tilde{v}(y, \alpha)\right] d\alpha < 0.$$

The fact that $\tilde{b}$ is an equilibrium of $\widetilde{\mathcal{A}}$ gives

$$\int_y^x \left[\tilde{v}(y, \alpha) - \tilde{v}(\alpha, \alpha)\right] d\alpha \leqslant 0.$$

Summing up these two integrals, we obtain

$$\int_y^x \left[\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)\right] d\alpha < 0,$$

which contradicts Corollary 9. This contradiction establishes the result. □

In fact, $V(x)$ is strictly increasing:

Lemma 14 $x > y \Rightarrow V(x) > V(y)$.

Proof Suppose that there exist $x > y$ such that $V(x) = V(y)$. Then, the monotonicity of V (by the previous lemma) gives

$$\forall \phi \in [y, x],\, V(\phi) = V(x) = V(y). \tag{31}$$

Let $S' = \left\{s \in S : \tilde{b}(y) \leqslant b(s) < \tilde{b}(x)\right\}$. From (10), for all $s \in S'$, $P^b(s) \in [y, x]$. Then, (31) implies that $V(s) = V(x)$, for $s \in S'$. Assumption 5 requires that $\mu\left(S'\right) = 0$. Observe that $S' = A \backslash B$, where $A \equiv \left\{s \in S : b(s) < \tilde{b}(x)\right\}$ and $B = \left\{s \in S : b(s) < \tilde{b}(y)\right\}$. Then, $\mu(A) = \mu(B)$. However, from the definition of $\tilde{b}$ as the inverse of $\tilde{P}^b$, we have:

$$0 < x - y = \tilde{P}^b\left(\tilde{b}(x)\right) - \tilde{P}^b\left(\tilde{b}(y)\right) = (\mu(A))^{N-1} - (\mu(B))^{N-1},$$

which is a contradiction. □

Thus, we proved that $x = P^b(s') > P^b(s) = y$ implies $\bar{v}(s') = V(x) > V(y) = \bar{v}(s)$ and $P^b(s') = P^b(s)$ implies $\bar{v}(s') = \bar{v}(s)$. In other words, $P^b(s') \lesseqgtr P^b(s)$ if and only if $\bar{v}(s') \lesseqgtr \bar{v}(s)$ which allows us to conclude that

$$P^b(t_i) = \Pr\left\{t_{-i} \in T_{-i} = S^{N-1} : \bar{v}(t_j) < \bar{v}(t_i),\, j \neq i\right\},$$

as defined in (15). In other words, the reparametrization is unique.

Uniqueness In the previous step (necessity), we showed that the reparametrization is unique, which implies that $\tilde{v}$ is unique. If $\tilde{v}$ is continuous, Proposition 1 says that any equilibrium $\tilde{b}$ satisfies the payment expression (1). If there is only one $\tilde{b}$ that satisfies such an expression, then the equilibrium of $\widetilde{\mathcal{A}}$ is unique and, hence, of $\mathcal{A}$.

Proof of Theorem 6 If $\tilde{b}$ is strictly increasing, then $b = \tilde{b} \circ P$ is an equilibrium of $\mathcal{A}$, by Theorem 5. Thus, we need to show that an equilibrium exists if $\tilde{b}$ is not increasing. In the first part of the proof of Theorem 5, we established that Assumption 5 implies that

$$\tilde{v}(x, z) > \tilde{v}(y, z), \forall z \in [0, 1]. \tag{32}$$

whenever $x > y$. It was also shown that

$$\int_y^x \left[\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)\right] d\alpha \geqslant 0 \tag{33}$$

and that $\tilde{\Pi}(P(t_i), c) = \Pi(t_i, c)$ for all t_i and c, when the reparametrization is given by (15).

Let us define $\overline{b}(x) = \sup_{\alpha \in [0,x]} \tilde{b}(\alpha)$. As we discussed after the statement of Theorem 3, this is just one of the possible specifications for the equilibrium bidding function. The only exception is when the tie includes the highest bidder. In such a case it is mandatory that the bid of tying bidders follows the above definition. The reason will become clear in the sequel.

Remember that $\tilde{b}$ is absolutely continuous. Then, there is an enumerable set of intervals $[a_k, c_k]$ where $\overline{b}(x)$ is constant. Let $b_k \equiv \overline{b}(x)$ for $x \in [a_k, c_k]$ (see Fig. 2). Therefore, there is a tie among types in $[a_k, c_k]$ for the bidding function $\overline{b}$. Let b_k be the specified bid for types in $[a_k, c_k]$, that is, $\overline{b}([a_k, c_k]) = \{b_k\}$. The tie is solved by an all-pay auction among tying bidders.

The only information that bidders have for the second auction is that there is a tie in b_k, that is, $P_{(-i)}(t_{-i}) \in [a_k, c_k]$.

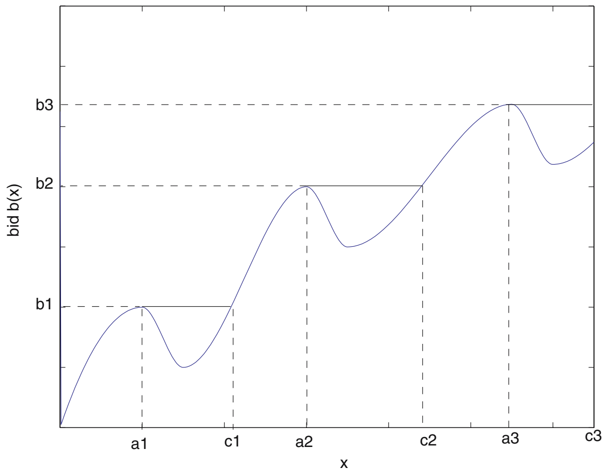

Fig. 2 Definition of $\overline{b}\,(x)$

By the definition of P in (15), $P_{(-i)}$ satisfies

$$\Pr\left(\left\{t_{-i} \in S^{N-1} : P_{(-i)}\,(t_{-i}) < x\right\} | P_{(-i)}\,(t_{-i}) \in [a_k, c_k]\right) = \frac{x - a_k}{c_k - a_k}.$$

Thus, in the tie-breaking auction the (direct) type t_i of bidder i is competing against players t_j in the set $\{s \in S : P\,(s) \in [a_k, c_k]\}$ and the equilibrium is to bid the increasing function

$$\tilde{b}^2\,(x) = \frac{1}{c_k - a_k}\int_{a_k}^{x} \tilde{v}\,(\alpha, \alpha)\,d\alpha.$$

Indeed, from (33), we have that

$$\frac{1}{c_k - a_k}\left[\int_{a_k}^{x} \tilde{v}\,(x, \alpha)\,d\alpha - \int_{a_k}^{x} \tilde{v}\,(\alpha, \alpha)\,d\alpha\right] \geqslant \frac{1}{c_k - a_k}\left[\int_{a_k}^{y} \tilde{v}\,(x, \alpha)\,d\alpha - \int_{a_k}^{y} \tilde{v}\,(\alpha, \alpha)\,d\alpha\right]$$

for any $x, y \in [a_k, c_k]$.

In the whole auction, the bidder with indirect type $x \in [a_k, c_k]$ who follows the strategy $\overline{b}\,(x)$ and $\tilde{b}^2(x)$ in the case of a tie has the expected payoff

$$\int_0^{a_k} \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha + (c_k - a_k)\left\{\frac{1}{c_k - a_k}\int_{a_k}^{x} \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha\right\}$$
$$= \int_0^{x} \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha.$$

We claim that deviation in the second auction is suboptimal. Deviating from $\overline{b}$ but bidding in the range of $\overline{b}$, that is, bidding $\overline{b}(y) \neq \overline{b}(x)$, yields

$$\tilde{\Pi}_i\left(x, \overline{b}(y)\right) = \int_0^{y} \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha,$$

when $\overline{b}(y)$ is not a bid with positive probability. This is lower than or equal to $\tilde{\Pi}_i\left(x, \overline{b}(x)\right)$ by (33). If $\overline{b}(y)$ is a bid with positive probability, the second stage will again be an all-pay auction and the bidder cannot improve his payoff, again by (33).

If x bids $\beta < \inf_{x\in[0,1]} \overline{b}(x)$, then his payoff will be

$$\int_0^1 p^L\left(\beta, \overline{b}(\alpha)\right) d\alpha \leqslant 0,$$

because $p^L \leqslant 0$. Therefore, this deviation cannot be profitable as well.

Finally, if x bids $\beta > \sup_{x\in[0,1]} \overline{b}(x) = \tilde{b}(\overline{x}) \geqslant \tilde{b}(1)$, for some $\overline{x}$. Since $\partial_1 p^W(\cdot) \geqslant 0$, $p^W\left(\beta, \overline{b}(z)\right) \geqslant p^W\left(\tilde{b}(1), \overline{b}(z)\right)$ and

$$\int_0^1 p^W\left(\beta, \overline{b}(\alpha)\right) \geqslant \int_0^1 p^W\left(\tilde{b}(1), \overline{b}(\alpha)\right) d\alpha = \int_0^1 \tilde{v}(\alpha,\alpha)\, d\alpha.$$

Then, the payoff of the bidder with type x who bids β is

$$\int_0^1 \left[\tilde{v}(x,\alpha) - p^W\left(\beta, \overline{b}(\alpha)\right)\right] d\alpha \leqslant \int_0^1 \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha$$
$$= \int_0^x \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha + \int_x^1 \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha$$
$$< \int_0^x \left[\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)\right] d\alpha,$$

where the last inequality comes from (32). Thus, deviation to β is unprofitable and $\tilde{b}$ is an equilibrium of the auction $\widetilde{\mathcal{A}}$.

From the fact that $\tilde{\Pi}\left(P\left(t_i\right), c\right)=\Pi\left(t_i, c\right)$ for all t_i and c, proved in Theorem 5, $b=\tilde{b} \circ P$ is an equilibrium of $\mathcal{A}$. □

Example 1 (proofs)

Suppose that a bidder bids β and the opponents follow

$$b(s)=\begin{cases}\frac{3\left(12+3s-2s^2\right)}{16}, & \text{if } s \in \left[0, \frac{1}{2}\right] \\ \frac{3\left(13+s-2s^2\right)}{16}, & \text{if } s \in \left(\frac{1}{2}, 1\right].\end{cases}$$

Let $g^1(\beta) \in \left[0, \frac{1}{2}\right]$ and $g^2(\beta) \in \left(\frac{1}{2}, 1\right]$ denote the types that bid β according to $b(\cdot)$. It is not difficult to see that $g^1(\beta)+g^2(\beta)=1$. The payoff is then

$$\begin{aligned}\Pi(x, b(s)) &= \int_0^{g^1(\beta)} [v(x, y)-\beta]\, dy + \int_{g^2(\beta)}^1 [v(x, y)-\beta]\, dy \\ &= \left(g^1(\beta)+1-g^2(\beta)\right)\left[\frac{9(x+1)}{8}+\frac{9\left(1-x^2\right)}{8}-\beta\right].\end{aligned}$$

If we denote $g^1(\beta)=s$ such that $g^2(\beta)=1-s$, we have the following first-order condition on s :

$$\partial_s \Pi=2\left[\frac{9\left(2+x-x^2\right)}{8}-\frac{3\left(12+3s-2s^2\right)}{16}\right]-2s\frac{3(3-4s)}{16}=0,$$

that is, $\frac{9\left(2+x-x^2\right)}{4}=\frac{9\left(2+s-s^2\right)}{4}$. The second-order derivative is

$$\partial_{ss}^2 \Pi=\frac{-9+12s-9+12s}{8}<0,$$

because $s<1/2$. This concludes the proof of the optimality of $b(\cdot)$.

Example 4 (proofs) We provide conditions on $\alpha(x)$, satisfied by the function specified in Example 4, such that there is a U-shaped equilibrium. In this case, there are two pooling types, that is, types which bid the same for each equilibrium bid. Thus, the pooling type of t, $\varphi=\varphi(t)$, in a symmetric equilibrium $b \in \mathcal{S}$ satisfies the condition of Proposition 3:

$$t+\alpha(t)E[t_2|b(t)=b(t_2)]=\varphi+\alpha(\varphi)E[t_2|b(t)=b(t_2)].$$

Since $E[t_2|b(t)=b(t_2)]=(t+\varphi)/2$ and because of the symmetry and the uniform distribution, then φ is the implicit solution of

$$(t+\varphi)(\alpha(\varphi)-\alpha(t))=2(t-\varphi).$$

The function $\alpha(x)$ defined in Example 4 satisfies the conditions of the following claim which establishes the existence of a symmetric U-shaped equilibrium.

Claim Suppose that: (i) α is differentiable, decreasing and convex such that $\alpha(0) - \alpha(1) = 2$ and (ii) α' is convex and $\alpha'(x) \geq -1/x$, for all $x \in (0, 1]$. Then there exists a U-shaped symmetric equilibrium.

Proof Define the following reparametrization:

$$P(t) = \frac{\varphi(t) - t}{2}.$$

It is easy to see that P is decreasing. Define

$$\begin{aligned}\widetilde{v}(x, y) &\equiv E\left[v(t_1, t_2) | P(t_1) = x, P(t_2) = y\right] \\ &= \frac{x + \varphi(x)}{2} + \frac{\alpha(x) + \alpha(\varphi(x))}{2} \frac{y + \varphi(y)}{2}.\end{aligned}$$

The specified bidding function is an equilibrium if $(x - y)\left[\widetilde{v}(x, y) - \widetilde{v}(y, y)\right] \geqslant 0$. Suppose that $x > y$. The last inequality divided by $(y + \varphi(y))/2$ is

$$\frac{x + \varphi(x)}{y + \varphi(y)} + \frac{\alpha(x) + \alpha(\varphi(x))}{2} \geqslant 1 + \frac{\alpha(y) + \alpha(\varphi(y))}{2}. \tag{34}$$

For each $w \in (0, 1]$, define $g_w(z) = \frac{z}{w} + \alpha(z)$, for $z \in [w, 1]$. It is easy to see that g is non-decreasing because $g_w'(z) = \frac{1}{w} + \alpha'(z) \geqslant \frac{1}{w} - \frac{1}{z} \geqslant 0$. Fix y and $w = \frac{y+\varphi(y)}{2}$. Since α is convex and $x \geqslant y$,

$$\begin{aligned}\frac{x + \varphi(x)}{y + \varphi(y)} + \frac{\alpha(x) + \alpha(\varphi(x))}{2} &\geqslant \frac{x + \varphi(x)}{y + \varphi(y)} + \alpha\left(\frac{x + \varphi(x)}{2}\right) \\ &= g_w\left(\frac{x + \varphi(x)}{2}\right),\end{aligned}$$

if $x + \varphi(x) \geqslant y + \varphi(y)$. Then, (34) is true if $t + \varphi(t)$ is non-increasing on t or, equivalently, $\varphi'(t) \leq -1$.

The implicit derivative of φ with respect to t is

$$\varphi'(t) = \frac{\alpha(t) - \alpha(\varphi(t)) + (t + \varphi(t))\alpha'(t) + 2}{\alpha(\varphi(t)) - \alpha(t) + (t + \varphi(t))\alpha'(\varphi) + 2}.$$

Without loss of generality (because $\varphi \circ \varphi(t) = t$), we can assume that the denominator is negative and $\varphi > t$. Thus,

$$\varphi'(t) \leqslant -1 \Leftrightarrow \frac{\alpha'(t) + \alpha'(\varphi)}{2} \geqslant \frac{\alpha(\varphi) - \alpha(t)}{\varphi - t}.$$

Since α' is a convex function, the above inequality holds if

$$\alpha'\left(\frac{t+\varphi}{2}\right) \geq \frac{\alpha(\varphi)-\alpha(t)}{\varphi-t} = -\frac{2}{t+\varphi}$$

where the last equality comes from the implicit definition of φ. However, this inequality is true because $\alpha'(x) \geq -1/x$, for all $x \in (0, 1]$. □

References

Araujo, A., de Castro, L.: Pure strategy equilibria of single and double auctions with interdependent values. Gam Econ Behav (forthcoming) (2007)

Araujo, A., Moreira, H.: Adverse selection problems without the Spence-Mirrlees condition, working paper, IMPA (2000)

Araujo, A., Moreira, H.: Non-monotone insurance contracts and their empirical consequences, working paper, IMPA (2003)

Athey, S., Levin, J.: Information and competition in U.S. Forest service timber auctions. J Polit Econ **109**, 375–417 (2001)

de Castro, L.: Affiliation, equilibrium existence and the revenue ranking of auctions, working paper 07-49 (25), Universidad Carlos III de Madrid, available at http://www.impa.br/~luciano/affiliation.pdf (2007)

de Frutos, M.A., Pechlivanos, L.: Second-price common-value auctions under multidimensional uncertainty. Gam Econ Behav **55**, 43–71 (2006)

Dasgupta, P., Maskin, E.: Efficient auctions. Q J Econ **115**, 341–388 (2000)

Ewerhart, C., Fieseler, K.: Procurement auctions and unit-price contracts. Rand J Econ **34**, 569–581 (2003)

Fullerton, R.L., McAfee, R.P.: Auctioning entry in tournaments. J Polit Econ **107**, 573–605 (1999)

Jackson, M.O., Simon, L.K., Swinkels, J.M., Zame, W.R.: Communication and equilibrium in discontinuous games of incomplete information. Econometrica **70**, 1711–1741 (2002)

Jackson, M.O., Simon, L.K., Swinkels, J.M., Zame, W.R.: Corrigendum to communication and equilibrium in discontinuous games of incomplete information. Econometrica **72**, 1927–1929 (2004)

Jehiel, P., Moldovanu, B., Stacchetti, E.: How (not) to sell nuclear weapons. Amer Econ Rev **86**, 814–829 (1996)

Kallenberg, O.: Foundations of Modern Probability. Springer Series in statistics. Heidelberg: Springer (2002)

Lehmann, E.L.: Testing Statistical Hypotheses. Wiley Series in Probability and Mathematical Statistics. New York: Wiley (1959)

Maskin, E., Riley, J.: Optimal auctions with risk averse buyers. Econometrica **52**, 1473–1518 (1984)

Maskin, E., Riley, J.: Equilibrium in sealed high bid auctions. Rev Econ Stud **67**, 439–454 (2000)

Mertens, J.F., Zamir, S.: Formulation of Bayesian analysis for games with incomplete information. Int J Game Theory **14**, 1–29 (1985)

Milgrom, P.R., Weber, R.J.: A theory of auctions and competitive bidding. Econometrica **50**, 1089–1122 (1982)

Monteiro, P.: First-price auction symmetric equilibria with a general distribution, working paper EPGE n. 568 (2004)

Reny, P., Zamir, S.: On the existence of pure strategy monotone equilibria in asymmetric first-price auctions. Econometrica **72**, 1105–1125 (2004)

Simon, L.K., Zame, W.R.: Discontinuous games and endogenous sharing rules. Econometrica **58**, 861–872 (1990)

Taylor, C.: Digging for golden carrots: an analysis of research tournaments. Am Econ Rev **85**, 872–890 (1995)

Zheng, C.Z.: High bids and broke winners. J Econ Theory **100**, 129–171 (2001)