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RESEARCH ARTICLE

Equilibrium existence and approximation of regular discontinuous games

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Abstract Many conditions have been introduced to ensure equilibrium existence in games with discontinuous payoff functions. This paper introduces a new condition, called regularity, that is simple and easy to verify. Regularity requires that if there is a sequence of strategies converging to s^* such that the players' payoffs along the sequence converge to the best-reply payoffs at s^*, then s^* is an equilibrium. We show that regularity is implied both by Reny's better-reply security and Simon and Zame's endogenous sharing rule approach. This allows us to explore a link between these two distinct methods. Although regularity implies that the limits of ϵ-equilibria are equilibria, it is in general too weak for implying equilibrium existence. However, we are able to identify extra conditions that, together with regularity, are sufficient for equilibrium existence. In particular, we show how regularity allows the technique of approximating games both by payoff functions and space of strategies.

Keywords Discontinuous games · Weak payoff security · Better-reply security · Regularity · Regular games · Epsilon-equilibria · Approximating sequence · Approximating games

JEL Classification C72 · C73

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1 Introduction

Many classical problems in economics are modeled as games with a continuum of actions, but discontinuous payoffs. In this category are Bertrand's duopolistic competition, Hotelling's spatial competition, auctions and many other games. The discontinuities in these games pose a difficulty for establishing equilibrium existence. Given the importance of discontinuous games, such difficulties have been stimulating efforts towards weakening the sufficient conditions for (pure strategy) equilibrium existence.

In an innovative paper, Simon and Zame (1990), henceforth SZ, observe that many cases of discontinuities arise from the specification of a tie-breaking or sharing rule. Although there is a usual way of breaking ties (splitting the prize in equal proportions), this is not always the only natural sharing rule. To illustrate this point, they offer the example of two psychologists choosing locations on a portion of Interstate 5 running through California and Oregon. The relevant position is represented by a point in the interval [0, 4]; the California portion is represented by [0, 3] and the Oregon part by [3, 4]. There is a continuum of potential clients uniformly distributed along the Interstate and, as in the classical Hotelling's model, each client chooses the psychologist located closest to him. In Simon and Zame's example, the psychologists are constrained to be in their own state. In this game, the natural equilibrium seems to be for both to be on the border (point 3). However, the standard sharing rule (that splits in equal proportions the clients) does not support this choice as equilibrium. In fact, with this sharing rule the game does not have an equilibrium. SZ then propose that the sharing rule is modified to reflect the limit of the proportion of clients (the psychologists' payoffs) from strategies that approximate the point 3, but that are not in a tie. That is, the Californian psychologist gets 3/4 of the patients if both choose to be in the border. In this way, one obtains a sharing rule under which there is an equilibrium.

In the important case of auctions, the use of special tie-breaking rules goes back at least to Lebrun (1996), who provides an example of an asymmetric private-value first-price auction with mass points such that no equilibrium exists with the standard tie-breaking rule. He then defines an augmented first-price auction where bidders are required to send a message together with their bids and shows equilibrium existence with this rule.[1] Maskin and Riley (2000) used a "second price auction tie-breaking rule", which consisted of running a second price auction in case of a tie. Jackson et al. (2002), henceforth JSSZ, provided an example of a symmetric interdependent-values auction where no equilibrium exists under the standard tie-breaking rule. From this, they extended SZ's ideas to games with incomplete information. Jackson and Swinkels (2005) applied this method of proof for establishing equilibrium existence in multi-unit private value auctions.[2] Araujo et al. (2008) showed that special tie-breaking rules may also be necessary when types are multidimensional and utilities are non-monotonic, even in the symmetric case. They showed equilibrium existence under the all-pay auction tie-breaking rule, which consisted in running an all-pay auction

[1] Lebrun (1996) also states a general theorem (his Theorem 3) for games with discontinuous payoffs, whose assumptions have some relation with Reny (1999)'s.

[2] Since they worked with private values with no mass points, they were able to prove that the definition of the tie-breaking rule was not important.

as the tie-breaking mechanism. Araujo and de Castro (2009) considered asymmetric single and double auctions, and showed that special tie-breaking rules are necessary in general. They were able to show that monotonic tie-breaking rules are sufficient for equilibrium existence.

A usual criticism of SZ's approach relies on its insistence on the *endogenous* definition of the sharing rule. This problem was explicitly indicated by Reny (1999, p. 1050): "in a mechanism design environment where discontinuities are sometimes deliberately introduced (auction design, for example), the participants must be presented with a game that fully describes the strategies and payoffs. One cannot leave some of the payoffs unspecified, to somehow be endogenously determined. In addition, this method is only useful in establishing the existence of a mixed, as opposed to pure, strategy equilibrium." However, these two shortcomings are not essential to the "special tie-breaking rule" approach, broadly defined.[3] Indeed, Araujo et al. (2008) showed that a special, but *exogenously* specified tie-breaking rule (the all-pay auction tie-breaking rule) is enough to guarantee equilibrium existence in a class of discontinuous games. Furthermore, both Araujo et al. (2008) and Araujo and de Castro (2009) present results in pure strategy equilibrium.

Another approach to equilibrium existence in discontinuous games was developed by Reny (1999) (see also Simon 1987; Dasgupta and Maskin 1986; Baye et al. 1993).[4] This approach is based on the better-reply security condition, which roughly requires that whenever a point is not equilibrium, one player can secure a payoff above her limit of payoffs, even if other players are allowed to slightly change their actions. This method seems to have absolutely no connection with the "special tie-breaking rule" approach described above. However, Jackson and Swinkels (2005) noticed that there are, indeed, a deep connection between these two methods:

> It is interesting that the tricky part of the proof using better-reply-security is to get a handle on the u^*'s in the closure of the game. The fact that they are those generated by omniscient tie-breaking suggests a deeper connection between the machinery of Reny and that of JSSZ. That is, a proof of existence via "apply JSSZ and check that some equilibria correspond to nice tie-breaking" and "check better-reply-security" are closely related. Because of the requirement that better-reply-security apply relative to all points in the closure of the graph, rather than just the graph, one has to understand exactly what might be in that closure; and the points in the closure are precisely the points that come from omniscient choices at points of discontinuity. On the other hand, in applying JSSZ, one has to understand the equilibria that might be generated under omniscient choices at points of discontinuity. In the auction setting, these two tasks are closely related.

[3] Reny (1999)'s comments were fair, nevertheless, since these two shortcomings were essential in SZ and also in Jackson et al. (2002).

[4] There is yet a third approach to equilibrium existence in discontinuous games, which uses fixed point theorems that take advantage of a natural order in the space of strategies (see for instance Vives 1990; Milgrom and Roberts 1990; Athey 2001; McAdams 2003; Fudenberg et al. 2007; Araujo and de Castro 2009). Although we will not discuss much this third approach, our regularity condition seems important for this method to apply.

How these approaches turn out to be related and which might be more efficient in other settings is an *open question*.[5]

This paper was motivated by this open question. We identify a condition, named *regularity*, that is implied by both Reny's and SZ's methods.[6] Whether regularity is or not a solution to the above question depends, of course, on the interpretation of what should be considered a relation between the two methods and since this is somewhat subjective, we leave to the reader to judge on this.[7] However, we illustrate how regularity plays an important role in both approaches.

Regularity is simple, easy to verify and is satisfied for most games with equilibria. Regularity requires that if there is a sequence of strategies converging to s^* such that the players' payoffs along the sequence converge to the best-reply payoffs at s^*, then the payoffs at s^* are equal to these limits, that is, s^* is an equilibrium (see Definition 3.1 for a formal statement). This captures the idea that the payoffs even at discontinuity points should be equilibrium payoffs, if there is a sequence of "almost" optimal points converging to it. Note that if the utility functions are continuous this is always the case, although regularity is much weaker than continuity.

By analyzing some examples that do not have equilibrium but satisfy other standard assumptions, we show that the failure of regularity may explain the failure of equilibrium existence in many cases. This is helpful, because it can indicate what should be the adaptation in the standard sharing rule of some games (such as the tie-breaking rules in auctions) necessary to ensure equilibrium existence.

On the other hand, regularity is too weak for guaranteeing equilibrium existence. However, under some extra-conditions (which amount to some kind of upper-semicontinuity as we will discuss later), regularity implies equilibrium existence. For instance, whenever one has ϵ-equilibria for all $\epsilon > 0$, regularity implies equilibrium existence (Theorem 4.1). In particular, if the value function is continuous,[8] then there exists an equilibrium (Corollary 4.3).

Instead of working with ϵ-equilibria, we can consider approximating games. Since a game is characterized by a pair of entities for each player—the strategy space and the

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[5] Jackson and Swinkels (2005, p. 121). The emphasis is ours.

[6] Regularity is used for many different mathematical concepts. For instance, we have regular topological spaces, regular measures, regular probability spaces, etc. Thus, the word is not very informative on its own, but do convey a single idea: if the concept in question is not *regular*, then weird behavior or properties are expected. That is, "regular" just means typical or canonical. This is exactly the idea that we want to convey with our regularity condition. Moreover, we were not able to find a more informative name, which was not too long or awkward.

[7] It is transparent from the above quote that the "open question" refers to *understanding* what is the relation between the methods. Thus, we believe that regularity is a solution to that open question exactly because it is a *simple* requirement that allows us to understand what both methods are doing with the points in the closure of the graph.

[8] The value function is the supremum of payoffs that a player can achieve, given the strategies by the others. See formal definition in the beginning of Sect. 3.

payoff function—there are two ways to approximate a game. The first one is to consider a sequence of payoff functions; the second is to consider a sequence of restricted strategy spaces.[9]

In the first case, the sequence of continuous functions is required to approximate the original function in a sense defined in Sect. 4.2, which is also implied by weak payoff security.[10] We show (Proposition 4.6) that the weak payoff security implies the existence of an approximating sequence of continuous functions. Then, if this approximating sequence is sufficiently well-behaved (not too much above the original payoff function), then compact and regular games have an equilibrium (Theorem 4.8). Finally, we show (Theorem 4.11) that regularity and lower-semicontinuity of the payoff function are sufficient for equilibrium existence if there is a sequence of games whose space of allowed actions approximates the original game, in a sense formalized in Sect. 4.3.

The technique of approximating discontinuous games by sequence of continuous ones is clearly not new. It probably goes back to the first attempts to prove equilibrium existence in discontinuous games. Also, the results of this paper are closely related to those obtained by Prokopovych (2010) and Carmona (2010b). We discuss the relation with these papers in Sect. 7.

The rest of this paper is organized as follows. In Sect. 2 we describe the basic setup and introduce the notation. Regularity is introduced in Sect. 3, which also discusses its basic properties. Section 4 collects our equilibrium existence results, while Sect. 5 illustrates the assumptions with some examples. Section 6 discusses the endogenous sharing rule method and clarifies the relation with Reny's method. A review of related literature is to be found in Sect. 7 and a conclusion, in Sect. 8.

2 Preliminaries

Let $I = \{1, \ldots, N\}$ be the set of players. Each player chooses a strategy from a compact convex subset S_i of a locally convex Hausdorff topological vector space.[11] Actually, we will implicitly assume that each S_i is metric, so that we can talk about sequences instead of nets. This restriction, however, is made only for simplicity, since the results and proofs also hold in the previously mentioned general setting. We summarize the profile of strategies by $s = (s_i, s_{-i}) \in S = S_i \times S_{-i}$, where $S_{-i} = \prod_{j \neq i} S_j$.

[9] Of course, it would be possible to have the two kinds of approximations at the same time, as we comment at the end of Sect. 4.

[10] Payoff security was introduced by Reny (1999) to characterize better-reply security and requires that any player is able to choose a single strategy and yet ensures that his payoff is at least $v_i(s_i, s_{-i}) - \epsilon$, even if the opponents choose strategies s_i' in a neighborhood of s_{-i}. Weak payoff security allows the player to choose different strategies for each s_{-i}' in the neighborhood of s_{-i}.

[11] A vector space is topological if it is endowed with a topology where the addition and multiplication by scalars are continuous transformations. A topological vector space is said to be locally convex if it possesses a base for its topology consisting of convex sets. This setup is slightly more restrictive than the one considered by Reny (1999).

Naturally, we endow S with the product topology. Since each S_i is compact, S is also compact, by Tychonoff theorem.

The payoff of player i is given by the function $v_i : S \to \mathbb{R}$, bounded above.[12] Occasionally, we will refer to $v : S \to \mathbb{R}^N$, understanding that the ith coordinate of $v(s)$ is denoted as $v_i(s)$. We denote the game by $(S_i, v_i)_{i\in I}$ but occasionally it will be convenient to refer to it only as v.

We say that v is quasiconcave if the sets $\{s_i \in S_i : v_i(s_i, s_{-i}) \geq \alpha\}$ are convex for all $i \in I$, $s_{-i} \in S_{-i}$ and $\alpha \in \mathbb{R}$. We say that v is compact if S is compact as described above. We denote the set of equilibrium points of $(S_i, v_i)_{i\in I}$ by $E(v)$, that is,

$$E(v) \equiv \left\{s \in S : v_i(s) \geq v_i(s_i', s_{-i}), \forall i \in I, s_i' \in S_i\right\}. \tag{1}$$

It will be convenient to define the best reply correspondence as follows:

$$\Gamma_v(s) \equiv \left\{\tilde{s} \in S : \forall i \in I, v_i(\tilde{s}_i, s_{-i}) = \sup_{s_i' \in S_i} v_i\left(s_i', s_{-i}\right)\right\}.$$

Of course, $s \in E(v)$ if and only if s is a fixed point of Γ_v.

3 Regular games

Given $v : S \to \mathbb{R}^N$, let us denote by $\hat{v} : S \to \mathbb{R}^N$ the function whose coordinates $\hat{v}_i : S_{-i} \to \mathbb{R}$ are the value functions, given by $\hat{v}_i(s_{-i}) \equiv \sup_{s_i' \in S_i} v_i(s_i', s_{-i})$, for each $i \in I$. Although $\hat{v}_i$ is a function only of s_{-i}, it will be convenient to abuse notation by considering $\hat{v}_i$ as a function of s. This shall not cause confusion. Of course, $v(s) \leq \hat{v}(s)$, where the inequality is in the coordinate-wise sense, that is, $v_i(s) \leq \hat{v}_i(s)$, for each $i \in I$. This function is used in the following:

Definition 3.1 Given $v : S \to \mathbb{R}^N$, the regularization of v is the function $\bar{v} : S \to \mathbb{R}^N$ defined by:

$$\bar{v}(s) = \begin{cases} \hat{v}(s), & \text{if } \exists s^n \to s \text{ such that } \lim_n v(s^n) \geq \hat{v}(s) \\ v(s), & \text{otherwise} \end{cases} \tag{2}$$

If $v(s) = \bar{v}(s)$, we say that v is regular at s. If v is regular at s for all $s \in S$, we simply say that v is regular.

The functions $v(s)$ and $\bar{v}(s)$ have different values only in the points that are not equilibrium, but would be equilibrium if the game was continuous. In fact, if v is continuous at s then $v(s) = \bar{v}(s)$. Now, regularity does not require v to be continuous, but if a point is a candidate to be equilibrium because there is a sequence of points approximating a candidate for equilibrium, then it must be an equilibrium. It is useful

[12] We use this assumption mainly for convenience. Since we are concerned with the points that maximizes the function, it is convenient that the value at this point is not infinite. As noted by Reny (1999), we can transform unbounded payoffs u_i in bounded ones, by adopting $v_i = \exp u_i/(1 + \exp u_i)$.

to observe that the regularization of a function is the same as the regularization of its regularization. In other words, we have the following:

Lemma 3.2 *If* $u : S \to \mathbb{R}^N$ *is the regularization of* $v : S \to \mathbb{R}^N$*, then* u *is regular.*

Proof First observe that $v(s) \leq u(s) \leq \hat{v}(s), \forall s \in S$. Therefore, $\hat{u}(s) = \hat{v}(s)$. Now, suppose that there is a sequence $\{s^n\}_{n\in\mathbb{N}}$ converging to s such that $\lim_n u(s^n) \geq \hat{u}(s)$. If $u(s^n) = v(s^n)$ for infinite many n, then $u(s) = \hat{v}(s) = \hat{u}(s) = \bar{u}(s)$. Otherwise, we can assume that $u(s^n) = \hat{v}(s^n) > v(s^n)$ for all n. This means that for each n, there exists a sequence $\{s^{n,m}\}_{m\in\mathbb{N}}$ such that $\lim_m s^{n,m} = s^n$ and $\lim_m v(s^{n,m}) \geq \hat{v}(s^n) = u(s^n)$. Since $\lim_n u(s^n) \geq \hat{v}(s)$, we can find a subsequence $\{x^j\}_{j\in\mathbb{N}}$ of $\{s^{n,m}\}_{n,m\in\mathbb{N}}$ that satisfies $x^j \to s$ and $\lim_j v(x^j) \geq \hat{v}(s)$. But this implies that $u(s) = \hat{v}(s) = \hat{u}(s) = \bar{u}(s)$. Therefore, u is regular. □

Regularity is implied by better reply security introduced by Reny (1999). To see this, let us recall some definitions. A player i can secure a payoff of $\alpha \in \mathbb{R}$ at $s \in S$ if there exists $s_i^* \in S_i$ such that $v_i(s_i^*, s_{-i}') \geq \alpha$ for all s_{-i}' in some open neighborhood of s_{-i}. A game $(v_i, S_i)_{i\in I}$ is better reply secure if whenever $(s^*, u^*) \in cl(\text{graph}(v))$, and s^* is not an equilibrium, some player i can secure a payoff strictly above u_i^* at s^*.

Proposition 3.3 *If* v *satisfies better reply security, then* v *is regular, that is,* $v = \bar{v}$.

Proof It is clear that $\bar{v} \geq v$. Suppose that $\bar{v}(s^*) = \hat{v}(s^*) > v(s^*)$. Thus, s^* is not an equilibrium point. By definition, there is $s^n \to s^*$ such that $u^* \equiv \lim_n v(s^n) \geq \hat{v}(s^*)$. Thus, $(s^*, u^*) \in cl(gr(v))$. By better reply security, there is a player $i \in I$, $s_i' \in S_i$, a neighborhood U of s_{-i}^* and $\delta > 0$ such that $v_i(s_i', \tilde{s}_{-i}) > u_i^* + \delta$ for all $\tilde{s}_{-i} \in U$. Since $s_{-i}^* \in U$,

$$u_i^* = \lim_n v_i(s^n) \geq \hat{v}_i(s^*) = \sup_{\tilde{s}_i \in S_i} v_i\left(\tilde{s}_i, s_{-i}^*\right) \geq v_i\left(s_i', s_{-i}^*\right) > u_i^* + \delta,$$

which is an absurd. The contradiction establishes that $\bar{v}(s^*) = \hat{v}(s^*) = v(s^*)$, that is, v is regular. □

Recall that $E(v)$ denotes the set of equilibrium points of v (see (1)). The following result clarify the relation between regularity and the existence of equilibrium:

Proposition 3.4 $E(v) = E(\bar{v}) \cap \{s : \bar{v}(s) = v(s)\}$.

Proof It is easy to see that whenever s is an equilibrium point, $v(s) = \bar{v}(s)$ and s is also an equilibrium of $\bar{v}$, that is, $E(v) \subset E(\bar{v}) \cap \{s : \bar{v}(s) = v(s)\}$. Now if $s \in E(\bar{v}) \cap \{s : \bar{v}(s) = v(s)\}$, then $v_i(s) = \bar{v}_i(s) \geq \bar{v}_i(s_i', s_{-i}) \geq v_i(s_i', s_{-i})$, for all $s_i' \in S_i$ and $i \in I$, which shows that $s \in E(v)$. □

Proposition 3.4 establishes that a point s is equilibrium for a game v if and only if s is equilibrium for the game $\bar{v}$ and v is regular at s. This implies that regularity is satisfied at all equilibria points, that is, regularity is "almost" necessary for equilibrium existence. We say "almost" because, as it turns out, a game may fail to be regular at some point $s \in S$, that is, $v(s) < \bar{v}(s)$, but the game has an equilibrium in another point s' (at which, the game will be regular). However, if the game is not regular, standard topological methods will need adaptations for ensuring equilibrium existence in such a game. The required adaptations and what we mean by "standard topological methods" are discussed below.

3.1 Topological methods and transfer conditions

By "standard topological methods," I loosely mean methods that find a sequence (or net) of strategies s^n (perhaps equilibrium of some approximating games v^n), use a compactness condition to find a subsequence converging to a point s^* and argue through some kind of continuity property that s^* is equilibrium. Many available approaches to equilibrium existence are topological in this sense, including Simon and Zame (1990), Reny (1999), Athey (2001) and Jackson et al. (2002).

However, one can see that a slight modification of the above described approach can still ensure equilibrium existence. Instead of requiring an assumption that would imply that that specific s^* is an equilibrium, it would be sufficient to conclude that some s' is an equilibrium. That is, one requires that the property holds not in the natural candidate point, but in some "transfer" point. Transfer conditions were introduced by Baye et al. (1993), but are also used in more recent papers, although not always explicitly (see for instance Prokopovych 2010; McLennan et al. 2009). Instead of discussing the use of this kind of assumption in other papers, we will limit ourselves to show how "transfer" conditions can be applied also to regularity, leading to a condition that is necessary for equilibrium existence.

Definition 3.5 A game $v : S \to \mathbb{R}^N$ is transfer-regular if the existence of a sequence $\{s^n\}_{n\in\mathbb{N}}$ converging to s and satisfying $\lim_n v(s^n) \geq \hat{v}(s)$ implies that there exists s' such that $v(s') = \hat{v}(s')$.

It is easy to see that if a game is regular then it is transfer-regular. We also have the following:

Proposition 3.6 *If a game $v : S \to \mathbb{R}^N$ has an equilibrium then it is transfer-regular, but the converse is not necessarily true.*

Proof Let s' be an equilibrium. Then, $v(s') = \hat{v}(s')$ and the game is automatically transfer-regular. To see that the converse does not hold, consider a game between two players, with $S_1 = S_2 = [0, 1]$, $v_1 : S \to \mathbb{R}$ given by:

$$v_1(s_1, s_2) = \begin{cases} -1, & \text{if } s_1 = s_2; \\ 1, & \text{if } s_1 \neq s_2; \end{cases}$$

and $v_2 : S \to \mathbb{R}$ given by:

$$v_2(s_1, s_2) = \begin{cases} 1, & \text{if } s_1 = s_2; \\ -1, & \text{if } s_1 \neq s_2; \end{cases}$$

(This is just a continuous version of matching pennies.) In this case, $\hat{v}(s) = (1, 1)$ while $v(s) \in \{(1, -1), (-1, 1)\}, \forall s \in S$. Therefore there is no sequence satisfying $\lim_n v(s^n) \geq \hat{v}(s)$, which shows that v is transfer-regular. □

All equilibrium existence results shown below are also valid for transfer-regular games, instead of just regular games. As we will see, regularity is used at the end of the proof, when we find a sequence $\{s^n\}_{n\in\mathbb{N}}$ converging to s such that $\lim_n v(s^n) \geq \hat{v}(s)$. At this point, we can use transfer-regularity instead of regularity to ensure equilibrium existence (see Corollaries 4.9 and 4.12). However, the paper is focused on regularity rather than transfer-regularity. There are two reasons for this. The first is for simplicity. The second is that it does not seem easier to check transfer-regularity than to check regularity itself, despite the fact that the former is weaker.

4 Pure strategy equilibrium existence

As we previously mentioned, regularity is not enough to guarantee equilibrium existence. See for instance the example contained in the proof of Proposition 3.6. In this section, we show how other conditions, together with regularity, ensure equilibrium existence.

4.1 Sequence of ϵ-equilibria

The first one is based on obtaining equilibrium through a sequence of ϵ-equilibria. The result is as follows:

Theorem 4.1 *If a game $(S_i, v_i)_{i\in I}$ is compact, regular, $\hat{v}$ is lower-semicontinuous and the game admits ϵ-equilibria for each $\epsilon > 0$, then it has an equilibrium.*[13]

Proof Suppose that s^n is a sequence of $\frac{1}{n}$-equilibria, that is, $\hat{v}(s^n) - \frac{1}{n} \leq v(s^n) \leq \hat{v}(s^n)$.[14] Compactness allows us to assume that $s^n \to s^*$ for some s^*, passing to a subsequence if needed. Since $\hat{v}$ is lower-semicontinuous, $\liminf_n \hat{v}(s^n) \geq \hat{v}(s^*)$. But we also have $\liminf_n v(s^n) \geq \liminf_n \left[\hat{v}(s^n) - \frac{1}{n}\right] = \liminf_n \hat{v}(s^n)$. Passing s^n to an appropriate subsequence (but still denoting this subsequence by s^n), we obtain $\lim_n v(s^n) \geq \hat{v}(s^*)$. Since v is regular, this implies that $v(s^*) = \bar{v}(s^*) = \hat{v}(s^*)$, that is, s^* is an equilibrium. □

Thus, Theorem 4.1 allows to obtain equilibrium existence under sufficient conditions for existence of ϵ-equilibria. For completeness, we state a set of sufficient conditions: the one provided by Prokopovych (2010). To state the result, we need the following definition, due to Reny (1999): the game $(S_i, v_i)_{i\in I}$ is payoff secure if for all $\epsilon > 0$, each player i can secure a payoff of $v_i(s) - \epsilon$ at s, that is, there exists $s'_i \in S_i$ and a neighborhood U of s_{-i} such that $v_i(s'_i, s_{-i}) > v_i(s) - \epsilon$ for all $s'_{-i} \in U$.

[13] Dasgupta and Maskin (1986) were the first to use lower-semicontinuity of $\hat{v}$.

[14] Recall that $v(s)$ and $\hat{v}(s)$ are vectors, so that $\hat{v}(s) - \frac{1}{n}$ is an abuse of notation, with the obvious meaning $\hat{v}(s) - (\frac{1}{n}, \ldots, \frac{1}{n})$. We will repeat this abuse of notation in other places.

Proposition 4.2 Prokopovych (2010) *If v is compact, quasiconcave, payoff secure and $\hat{v}$ is continuous, then it possesses a pure strategy ϵ-equilibrium for every $\epsilon > 0$.*

This gives the following:

Corollary 4.3 *If a game $(S_i, v_i)_{i\in I}$ is compact, quasiconcave, regular, payoff secure and $\hat{v}$ is continuous, then it has a pure strategy equilibrium.*[15]

4.2 Approximating payoff functions

A setting in which regularity is particularly useful occurs when there is a sequence of games approximating the original one, each of which has an equilibrium. In this case, regularity can be used to show that the limit of the equilibria of the approximating games is an equilibrium of the original game.

There are two ways to define approximating games: by approximating payoff functions and by approximating spaces. In this section we consider approximation of the payoff functions; next section deals with space approximation.

Definition 4.4 (*Approximating functions*) A sequence of continuous quasiconcave functions $\{v^n : S \to \mathbb{R}^N\}_{n\in\mathbb{N}}$ is an approximating sequence of the game $G = (S_i, v_i)_{i\in I}$ if it satisfies the following:

1. $v^n(s) \le \hat{v}(s)$ for all $s \in S$ and $n \in \mathbb{N}$;
2. If $s^n \to s^*$, then $\liminf_n v_i^n(s^n) \ge \hat{v}_i(s^*)$.

It is clear that if $\hat{v}$ is continuous then $v^n = \hat{v}$ defines an approximating sequence.[16] However, the existence of approximating sequences requires less than this. Consider the following definition, which is weaker than payoff security:

Definition 4.5 (*Weakly payoff secure*) We say that v is weakly payoff secure if for all $i \in I$, $\epsilon > 0$ and $s \in S$, there exists an open neighborhood U of s_{-i} such that for each $s'_{-i} \in U$, there exists $s'_i \in S_i$ such that $v_i(s'_i, s'_{-i}) \ge v_i(s) - \epsilon$.[17]

Weak payoff security implies the existence of an approximating payoff sequence:

Proposition 4.6 *Assume that v is weakly payoff secure and S is compact metric. Then, there exists a sequence of approximating functions of v.*

Proof First, let us show that $\hat{v}_i$ is lower-semicontinuous, that is, the set $\{s_{-i} \in S_{-i} : \hat{v}_i(s_{-i}) > \alpha\}$ is open for all $\alpha \in \mathbb{R}$ and $i \in I$. Indeed, fix s_{-i} in this set and choose $\epsilon > 0$ such that $\hat{v}_i(s_{-i}) - \epsilon > \alpha$. Recall that $\hat{v}_i(s_{-i}) = \sup_{\tilde{s}_i\in S_i} v_i(\tilde{s}_i, s_{-i})$. Then,

[15] This result is actually true without the assumption that $\hat{v}$ is continuous, as shown by Carmona (2010b)—see discussion in Sect. 7.

[16] Note that $\hat{v}$ is trivially quasiconcave.

[17] In other words, there is a non-empty correspondence $\varphi_i : U \to S_i$ that gives $s'_i \in \varphi_i(s'_{-i})$ for each $s'_{-i} \in U$ satisfying the required inequality. Note, however, that we do not require this correspondence to have convex values or be upper or lower-semicontinuous.

there exists $\tilde{s}_i$ such that $v_i(\tilde{s}_i, s_{-i}) > \hat{v}_i(s_{-i}) - \epsilon$. Since the game is weakly payoff secure, there exists open neighborhood $U_{s_{-i}}$ of s_{-i} such that:

$$\hat{v}_i(s'_{-i}) \geq v_i(s'_i, s'_{-i}) \geq v_i(\tilde{s}_i, s_{-i}) > \hat{v}_i(s_{-i}) - \epsilon, \forall s'_{-i} \in U_{s_{-i}}.$$

This shows that all s'_{-i} in U are also in $\{s_{-i} \in S_{-i} : \hat{v}_i(s_{-i}) > \alpha\}$, that is, $\hat{v}_i$ is lower-semicontinuous.

By Reny (1999), Lemma 3.5, there exists a sequence of continuous functions $v_i^n : S_{-i} \to \mathbb{R}$ satisfying conditions 1 and 2 of the definition of approximating payoff sequence. Since each v_i^n does not depend on s_i, it is quasiconcave. This concludes the proof. □

Now, consider the following:

Definition 4.7 (*function-approximated game*) We say that v is function- approximated if there is an approximating payoff sequence v^n such that, if s^n is a sequence of equilibria for $(S_i, v_i^n)_{i \in I}$ then for each $\epsilon > 0$, there exists n_ϵ such that $v^n(s^n) \leq v(s^n) + \epsilon$ for all $n \geq n_\epsilon$.

This condition guarantees that the approximating payoff sequence can be taken not too above v, at least for equilibrium points of the approximating game. With these definitions in place, we have the following:

Theorem 4.8 *If a game $(S_i, v_i)_{i \in I}$ is compact, regular and function-approximated, then it has a pure strategy equilibrium.*

Proof Take an approximating payoff sequence $v^n : S \to \mathbb{R}^N$ that function-approximates v. Since $(S_i, v_i^n)_{i \in I}$ is compact and quasiconcave and the function v^n is continuous, there exists a pure strategy equilibrium s^n. By compactness, we may assume (passing to subsequences if necessary) that $s^n \to s^*$. Since v^n is an approximating payoff sequence, $\hat{v}(s^*) \leq \liminf_n v^n(s^n)$. Using the fact that s^n is equilibrium for v^n, we have $v^n(s^n) = \hat{v}^n(s^n)$. Therefore, we can choose a subsequence (denoted again by s^n) such that

$$\hat{v}(s^*) \leq \liminf_n v^n(s^n) \leq \liminf_n \left(v(s^n) + \frac{1}{n} \right) = \liminf_n v(s^n),$$

where the second inequality holds because v is function-approximated. We can now pass to a subsequence, if necessary, to obtain $\hat{v}(s^*) \leq \lim_n v(s^n)$. Since v is regular, this implies that $v(s^*) = \hat{v}(s^*)$, that is, s^* is equilibrium. □

From the proof, we can see that function-approximation and regularity guarantee that the limit of equilibrium points in the approximating games is an equilibrium of the original game. After the discussion at the end of Sect. 3, we obtain the following:[18]

Corollary 4.9 *Let $G = (S_i, v_i)_{i \in I}$ be compact and function-approximated. Then G has a pure strategy equilibrium if and only if it is transfer-regular.*

[18] The converse comes directly from Proposition 3.6.

4.3 Approximating the strategy space

The second way of defining approximating games is through the set of strategies available for each player.

Definition 4.10 (*space-approximated game*) Given a game $G = (S_i, v_i)_{i \in I}$, a sequence of games $(S_i^n, v_i^n)_{i \in I}$ is an approximating-spaces sequence (for G) if it satisfies the following for each r:

1. $S_i^n \subset S_i$;
2. v_i^n is the restriction of v_i to $S^n \equiv \prod_i S_i^n$. Thus, we can write only v_i instead of v_i^n from now on;
3. the game $(S_i^n, v_i)_{i \in I}$ has an equilibrium, that is, there exists $s^n \in S^n$ such that $v_i(s^n) \geq v_i(x_i^n, s_{-i}^n)$, $\forall x_i^n \in S_i^n$.
4. $\forall x_i \in S_i$, there exists a sequence $\{x_i^n\}$ such that $x_i^n \in S_i^n$ and $x_i^n \to x_i$.

If G has an approximating-spaces sequence, then we say that G is space- approximated.

SZ approximate the original spaces S_i using finite sets S_i^n. However, this is not necessary for the definition above. The spaces S_i^n can be infinite sets with restricted available strategies, so that an equilibrium exists. We have the following:

Theorem 4.11 *Assume that a game $G = (S_i, v_i)_{i \in I}$ is compact, regular, space-approximated and lower-semicontinuous.*[19] *Then G has an equilibrium.*

Proof Let s^n be such that $v_i(s^n) \geq v_i(x_i^n, s_{-i}^n)$, for every $x_i^n \in S_i^n$. By compactness of S, s^n converges (passing to subsequences if needed), to some s^*. For each $x_i \in S_i$, fix a sequence $x_i^n \in S_i^n$, such that $x_i^n \to x_i$. Since v_i is lower-semicontinuous and $(x_i^n, s_{-i}^n) \to (x_i, s_{-i}^*)$,

$$\liminf_n v_i(s^n) \geq \liminf_n v_i(x_i^n, s_{-i}^n) \geq v_i(x_i, s_{-i}^*).$$

Since x_i was arbitrary, this implies that $\liminf_n v_i(s^n) \geq \hat{v}_i(s^*)$. Passing to a subsequence if needed, we have $\lim_n v_i(s^n) \geq \hat{v}_i(s^*)$, for all i. Since v is regular, s^* is an equilibrium. □

As Sect. 6 discusses, Theorem 4.11 is related to the argument used in the proof of SZ's main result. The following result is parallel to Corollary 4.9:

Corollary 4.12 *Let $G = (S_i, v_i)_{i \in I}$ be compact, space-approximated and lower-semicontinuous. Then G has a pure strategy equilibrium if and only if it is transfer-regular.*

A final remark is that we could in principle combine the ideas of approximation of payoff functions *and* of space of strategies at the same time. Since this would be a simple variation of the ideas above, we refrain from spelling out the correspondent details.

[19] We say that $G = (S_i, v_i)_{i \in I}$ is lower-semicontinuous if v_i is lower-semicontinuous, $\forall i \in I$.

5 Examples

In this section, we illustrate how the failure of equilibrium existence is related to the failure of regularity. The first example is example 1 of Carmona (2005):

Example 5.1 Let $I = \{1, 2\}$, $S_1 = S_2 = [0, 1]$, $v_1 : S \to \mathbb{R}$ given by:

$$v_1(s_1, s_2) = \begin{cases} 0, & \text{if } s_2 \le \frac{1}{2} - s_1; \\ 2, & \text{if } s_1 = 0 \text{ and } s_2 > \frac{1}{2}; \\ 1, & \text{otherwise} \end{cases}$$

and $v_2 : S \to \mathbb{R}$ given by:

$$v_2(s_1, s_2) = \begin{cases} 0, & \text{if } s_1 \le \frac{1}{2} \text{ and } s_2 > 0; \\ 1, & \text{if } s_1 \le \frac{1}{2} \text{ and } s_2 = 0; \\ 1, & \text{if } s_1 > \frac{1}{2} \text{ and } s_2 \le \frac{1}{2}; \\ 2, & \text{if } s_1 > \frac{1}{2} \text{ and } s_2 > \frac{1}{2}; \end{cases}$$

Carmona (2005, Proposition 1) shows that the game in Example 5.1 is quasiconcave and payoff secure, but has no pure strategy equilibrium or ϵ-equilibrium for $\epsilon > 0$ sufficiently small. It is not difficult to see that:

$$\hat{v}_1(s_2) = \begin{cases} 2, & \text{if } s_2 > \frac{1}{2}; \\ 1, & \text{otherwise} \end{cases}$$

and

$$\hat{v}_2(s_1) = \begin{cases} 1, & \text{if } s_1 \le \frac{1}{2}; \\ 2, & \text{if } s_1 > \frac{1}{2}; \end{cases}$$

so that $\hat{v}_i$ is not continuous for $i = 1, 2$. On the other hand, since the game is payoff secure, it is also weakly payoff secure and it has an approximating payoff sequence. However, it is not regular. To see this, observe that $\hat{v}(\frac{1}{2}, \frac{1}{2}) = (1, 1)$, but $\lim_n v(\frac{1}{2} + \frac{1}{n}, \frac{1}{2} + \frac{1}{n}) = (1, 2) \gneqq (1, 1) = \hat{v}(\frac{1}{2}, \frac{1}{2}) \gneqq (1, 0) = v(\frac{1}{2}, \frac{1}{2})$.

Now we consider example 3 of Prokopovych (2008).

Example 5.2 Let $I = \{1, 2\}$, $S_1 = S_2 = [0, 1]$, $v_1 : S \to \mathbb{R}$ given by:

$$v_1(s_1, s_2) = \begin{cases} 1 - s_1, & \text{if } s \in [0, 1] \times \{0\}; \\ 1 + s_1 s_2, & \text{if } s \in [0, 1] \times (0, 1]; \end{cases}$$

and $v_2 : S \to \mathbb{R}$ given by:

$$v_2(s_1, s_2) = \begin{cases} s_2, & \text{if } s \in \{0\} \times [0, 1]; \\ 1 + s_1(1 - s_2), & \text{if } s \in (0, 1] \times [0, 1]; \end{cases}$$

As Prokopovych (2008) observes, this game is compact, quasiconcave, payoff secure and $\hat{v}$ is continuous, but it does not have a pure strategy equilibrium. Let us verify that it is not regular. It is easy to see that:

$$\hat{v}_1(s_2) = \begin{cases} 1, & \text{if } s \in [0,1] \times \{0\}; \\ 1 + s_2, & \text{if } s \in [0,1] \times (0,1]; \end{cases}$$

and

$$\hat{v}_2(s_1) = \begin{cases} 1, & \text{if } s \in \{0\} \times [0,1]; \\ 1 + s_1, & \text{if } s \in (0,1] \times [0,1]; \end{cases}$$

and $\hat{v}(0,0) = (1,1) = \lim_n (1 + \frac{1}{n^2}, 1 - \frac{n-1}{n^2}) = \lim_n v(\frac{1}{n}, \frac{1}{n})$. However, $v(0,0) = (1,0) \neq (1,1) = \hat{v}(0,0)$, that is, regularity is not satisfied.

6 The endogenous sharing rule method

This section examines the "endogenous sharing rule" approach introduced by SZ and later developed by Jackson et al. (2002) and Jackson and Swinkels (2005). This approach's main ideas are also present in Araujo and de Castro (2009) and Araujo et al. (2008), although the last paper defines an explicit (special) sharing rule.

Let us begin by describing SZ's framework, which is slightly different from Reny's. After we understand SZ's framework, we can see how the two are related. Instead of a utility function $v : S \to \mathbb{R}^N$ as before, SZ consider a *payoff correspondence* $Q : S \to \mathbb{R}^N$. They interpret $Q(s)$ "as the universe of utility possibilities given the strategy profile s" (p. 864). They then assume that Q is bounded and upper-semicontinuous, with non-empty, convex, compact values.

A natural way to think about Q is as follows. Suppose that we have a game $(S_i, v_i)_{i \in I}$ as previously defined, where each v_i can be discontinuous. Say that a correspondence $P : S \to \mathbb{R}^N$ extends $v : S \to \mathbb{R}^N$ if v is a selection of P and that P is *standard* if it is bounded and upper-semicontinuous, with non-empty, convex, compact values. Now define $V : S \to \mathbb{R}^N$ as the smallest standard correspondence that extends v, that is, if P is standard and extends v then $V(s) \subset P(s)$ for every $s \in S$. Of course we have to establish that this definition is not vacuous. This comes from the following:

Lemma 6.1 *V is well defined.*

Proof Let $\mathcal{S}$ denote the set of standard correspondences $P : S \to \mathbb{R}^N$ which extend v. The above definition is equivalent to put $V(s) \equiv \cap_{P \in \mathcal{S}} P(s)$. Since $v(s) \in P(s)$, for all $s \in S$ and $P \in \mathcal{S}$, then $v(s) \in V(s)$, that is, V is non-empty and extends v. Moreover, it has convex compact values, since each $P \in \mathcal{S}$ has and arbitrary intersections preserve convexity, compactness and closedness. It remains to verify that it is upper-semicontinuous, which is equivalent to having a closed graph (since its image is in $\mathbb{R}^N$). However,

$$\begin{aligned}\operatorname{graph}(V) &= \{(s,u) \in S \times \mathbb{R}^N : u \in V(s) = \cap_{P \in \mathcal{S}} P(s)\},\\ &= \cap_{P \in \mathcal{S}} \{(s,u) \in S \times \mathbb{R}^N : u \in P(s)\}\\ &= \cap_{P \in \mathcal{S}} \operatorname{graph}(P).\end{aligned}$$

Since all $P \in \mathcal{S}$ have closed graphs, this concludes the proof. □

Thus, we can interpret the primitive correspondence Q in SZ's framework just as the correspondence V that could be defined in Reny's framework as above. This shows that the two setups are essentially interchangeable. To fix ideas, from now on, let Q be just the correspondence defined as a primitive by SZ.

As said above, SZ assume that Q is standard. For our purposes, the key property is that Q is upper-semicontinuous. According to SZ, this assumption "means that the set of utility possibilities for each strategy profile is at least as large as the set of limits of utility possibilities of nearby profiles." This already gives a sense on how our regularity condition will be related to SZ's approach. But before making this formal, we need to understand another important concept in SZ's framework: that of sharing rule; for this, we find nothing better than to quote SZ: "A *sharing rule* is a Borel measurable selection $q : S \to \mathbb{R}^N$ such that $q(s) \in Q(s)$ for each $s \in S$. Since $Q(s)$ is the universe of utility possibilities given the strategy profile s, a sharing rule is just a particular choice of payoff at each point of the space of strategy profiles" (p. 864).

Now, the way that the two approaches are related should be more or less clear. When Reny fixes a utility function $v : S \to \mathbb{R}^N$, he is already fixing the selection q of Q in SZ's terminology. When he requires that v satisfies some properties (in his case, better-reply security), this is akin to require that there exists a selection q of Q that has that property. If equilibrium can be proved for that selection, then an equilibrium with endogenous sharing rule exists.[20]

Proposition 6.2 *Let Q be a standard correspondence. Then there is a selection q of Q such that q is regular.*

Proof Let $\bar{Q}$ denote the sub-correspondence of Q formed by the maximal points with respect to the component-wise order in $\mathbb{R}^N$, that is,

$$\bar{Q}(s) \equiv \{y \in Q(s) : \forall x \in Q(s), x \geq y \Rightarrow x = y\}.$$

Since $Q(s)$ is non-empty, so $\bar{Q}(s)$ is. Let v be a selection of $\bar{Q}$. We claim that v is regular. For, let s^n be a sequence converging to s^* such that $y \equiv \lim_n v(s^n) \geq \hat{v}(s^*) \geq v(s^*)$. Since Q is upper-semicontinuous and $\{(s^n, v(s^n))\}_n$ is in the graph of Q, $y \in Q(s^*)$. Since $v(s^*) \in \bar{Q}(s^*) \subset Q(s^*)$, this means that $y = v(s^*)$, that is, v is regular at s^*. This completes the proof. □

[20] While Reny uses a space of strategies and the equilibrium is in pure strategy in that space, SZ work only with mixed strategy. One may think that this is a fundamental difference, but this is not quite true. First, SZ's idea can be adapted to work with pure strategies (see Araujo and de Castro 2009). Second, the strategy space in Reny's framework can already be the space of mixed strategies.

Therefore, as in the Reny's method, regularity is implied by SZ's assumptions. Also as before, the converse is not necessarily true, since regularity is obviously too weak to imply something about the upper-semicontinuity of Q.

Now consider SZ's framework with payoff correspondence Q and consider that each selection v of Q defines a game $G = (S_i, v_i)_{i \in I}$ as specified before. We say that Q has an endogenous sharing rule equilibrium if there is a selection v of Q such that G has an equilibrium. Then, the following is an immediate corollary of Theorem 4.11.

Corollary 6.3 *Assume that there is a selection v of Q such that $G = (S_i, v_i)_{i \in I}$ is compact, space-approximated, regular and lower-semicontinuous. Then Q has an endogenous sharing rule equilibrium.*

Now we review SZ's proof and show that it amounts, *essentially*, to verify the assumptions of Corollary 6.3 and to prove Theorem 4.11, although in a different form. Their proof is divided into six steps: (1) finite approximation—the game is approximated by finite games; (2) limits—the approximation games are taken to their limits; (3) selections; (4) better responses; (5) perturbation; (6) solution. The first step is the establishment of a sequence of approximating games $G^n = (S_i^n, v_i^n)_{i \in I}$ which have a (mixed strategies) Nash equilibrium $\alpha^n = (\alpha_1^n, \ldots, \alpha_N^n)$.[21] These games form a *space-approximating sequence*. Step 2 uses a kind of compactness property to show that a subsequence of these equilibrium profiles converges to a mixed strategy profile $\alpha = (\alpha_1, \ldots, \alpha_N)$ and an appropriate subsequence of payoff functions for the approximating games also converges to a payoff function q. Step 3 shows that q corresponds to a sharing rule, that is, a selection of Q. Step 4 proves that the set of strategies that are strictly better than the limit α is a zero-measure set. Since the strategies are mixed, this amounts to prove that q is *regular* at α.[22] Step 5 then modifies q by punishing the players that play in the set of strictly better alternatives to α, by establishing the lowest possible payoff at those points. Thus, this step constructs a perturbed sharing rule $\tilde{q}$ which is *lower-semicontinuous*. Finally, step 6 is essentially the argument used in the proof of Theorem 4.11, which directly leads to Corollary 6.3. As we can see, the proof verifies the assumptions of Corollary 6.3 (for a fixed equilibrium α): that $\tilde{q}$ is a selection of Q (step 3) that is space-approximated (step 1), compact (step 2), regular (step 4) and lower-semicontinuous (step 5).

Jackson et al. (2002) (JSSZ) extend SZ's result for games with incomplete information. For this, they require that players announce their types, obtaining an augmented game similar to the one used by Lebrun (1996). JSSZ's proof is similar to SZ's, consisting also of six steps, although there are some subtle and difficult differences. It is natural to expect that Theorem 4.11 and Corollary 6.3 could be similarly extend to deal explicitly with games of incomplete information, but since the adaptations were

[21] They use mixed strategies essentially for two reasons. The first one is just that the approximating games are finite. Second, this space of strategies is compact in its natural (weak) topology.

[22] The equilibrium α is also fixed in their proof, not only the sharing rule. This is perhaps the more important difference between their result and Corollary 6.3, but this is not essential, since the proof of Theorem 4.11 also works for a particular point.

already made clear by JSSZ and would require lengthy technical arguments, we refrain to undertake this extension here.[23]

7 Relation with other methods

This section discusses relation with the more recent literature. For a more comprehensive review of the literature (see Carmona 2010a).[24]

In a recent paper, Carmona (2010b) generalizes the pure strategy equilibrium existence results of Reny (1999) and Barelli and Soza (2009) for metric spaces. First, he defines a game $(S_i, v_i)_{i \in I}$ to be better-reply closed relative to a function $\underline{u} : S \to \mathbb{R}^N$ if s^* is an equilibrium whenever $(s^*, u^*) \in cl(\text{graph}(v))$ and $u_i^* \geq \hat{\underline{u}}_i(s_{-i}^*)$ for all $i \in I$. Maybe the weakest form of better-reply closeness occurs when the game $(S_i, v_i)_{i \in I}$ is better-reply close relative to v itself. This is actually equivalent to regularity, as the following lemma clarifies.

Lemma 7.1 *A game is regular* iff *it is better-reply closed relative to itself.*

Proof Sufficiency: let $u^* \equiv \lim_n v(s^n)$, for some sequence $s^n \to s^*$, which means that $(s^*, u^*) \in cl(\text{graph}(v))$. Since the game is better-reply closed relative to itself, then "$u_i^* \geq \hat{v}_i(s^*)$ for all i" implies that s^* is equilibrium. This means that regularity is satisfied.

Necessity: let $(s^*, u^*) \in cl(\text{graph}(v))$ be such that $u_i^* \geq \hat{v}_i(x_i^*)$. Since $(s^*, u^*) \in cl(\text{graph}(v))$, there is a sequence $\{s^n\}$ converging to s^* such that $v_i(s^n) \to u_i^*$ for all $i \in I$. By regularity, s^* is an equilibrium. □

Therefore, one of the results in Carmona (2010b) implies our Corollary 4.3: if G is compact, quasiconcave, regular and payoff secure, then G has a Nash equilibrium.

Carmona (2010b) says that $(S_i, v_i)_{i \in I}$ is *generalized payoff secure* if for all $i \in I$, $\epsilon > 0$ and $s \in S$, there exists an open neighborhood $V_{s_{-i}}$ of s_{-i} and a non-empty, closed, convex valued, upper- or lower-semicontinuous correspondence $\varphi_i : V_{s_{-i}} \rightrightarrows S_i$ such that $v_i(s') \geq v_i(s) - \epsilon$ for all $s' \in \text{graph}(\varphi_i)$. A game $(S_i, v_i)_{i \in I}$ is approximately payoff secure relative to $\underline{u}$ if, for all $i \in I$, $\underline{u}_i \leq v_i$, $\underline{u}_i$ is quasiconcave and $(S_i, \underline{u}_i)_{i \in I}$ is generalized payoff secure. As Carmona (2010b) shows in his Lemma 2, if $(S_i, v_i)_{i \in I}$ is approximately payoff relative to v itself, then $\hat{v}_i$ is lower-semicontinuous. Therefore, by the proof of our Lemma 4.6 this property is sufficient for the existence of an approximating payoff sequence (incidentally, weakly payoff secure is strictly weaker than generalized payoff secure). A close look at Carmona (2010b)'s proof suggests that the function-approximation property is also implied by his assumptions, although such assumptions refer to a function $\underline{u}$, which makes the comparison not straightforward. On the other hand, weakly reciprocal upper-semicontinuity (wrusc)

[23] In some sense, Corollary 6.3 already deals with games of incomplete information if we see the strategy spaces as function spaces from types to the original actions and the announced types. The only difference is that the specific strategy space, the notions of convergence and the aspects of allowing announcement of types are not explicitly considered in Corollary 6.3.

[24] I am extremely grateful to Guilherme Carmona for many comments relevant to this section, including the proof of the "only if" (necessity) part of Lemma 7.1.

implies regularity. This comes from the proof of Theorem 5 in Carmona (2010b), which establishes that a game is better-reply closed relative to itself (hence regular, by Lemma 7.1), if and only if it is wrusc at s for all s that are not equilibrium.

Another relevant paper is that of Prokopovych (2010). It is possible to say that the three papers (this one, Prokopovych 2010; Carmona 2010b) make a similar point: that a condition akin to lower-semicontinuity (either payoff security or approximation of the game) suffices to obtain a sequence of strategies that are "almost-equilibria." Then, an upper-semicontinuity property like weak reciprocal upper-semicontinuity or regularity implies that the limit of the sequence of "almost-equilibria" is an equilibrium. Although similar, the approaches also have some differences. In the other two papers, there is no approximation of the game, rather one obtains directly a sequence of generalized approximate equilibria as a consequence of payoff security. In contrast, in this paper, we consider directly the approximation of the game in two forms: approximation of the payoff functions and approximation of the space of strategies. We also emphasized one condition (regularity) over the companion lower-semicontinuity-type conditions, because of our primary purpose to establish a link between Reny and SZ's methods. As it turns out, the lower-semicontinuity strategy in these two results are relatively different, while regularity is common to both.

In another recent paper, Reny (2009) introduces a new condition, called lower single-deviation property, and proves that it generalizes better-reply secure and it is sufficient for equilibrium existence. This condition also implies regularity.

8 Conclusion

This paper offered a new assumption, regularity, which is both simple, easy to verify and central to equilibrium existence. Many examples that fail to have equilibrium, fail precisely because of the failure of satisfying regularity. However, regularity is too weak for being sufficient for equilibrium existence. We provide extra conditions under which one can ensure equilibrium existence.

As we have argued, regularity is a property implied both by better-reply security and by SZ's approach. Jackson and Swinkels (2005) have previously noted the connection between the endogenous tie-breaking method and Reny's better-reply, but they left open the understanding what both methods have in common: "How these approaches turn out to be related and which might be more efficient in other settings is an open question." (p. 121) This paper contributes to clarify and understand the connection between these different methods.

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