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EQUILIBRIA EXISTENCE AND CHARACTERIZATION IN AUCTIONS: ACHIEVEMENTS AND OPEN QUESTIONS

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Abstract. Establishing existence and characterizing equilibria are both important achievements in the study of auctions. However, we recognize that equilibria existence results form the basis for well accepted characterizations. In this survey, we review the landmark results and highlight open questions regarding equilibria existence and characterizations in auctions. In addition, we review the standard assumptions underlying these results, and discuss the suitability of the Nash equilibrium solution concept. We focus our review on single-object auctions, but also review results in multi-unit, divisible, combinatorial and double auctions.

Keywords. Equilibrium existence; Multi-unit auctions; Single-unit auctions

1. Introduction

The widespread use of auctions creates strong demand for results both to inform practitioners and to advance the theoretical understanding of auctions. Although many textbooks and surveys have been written,[1] the literature on equilibria existence and characterization in auctions still lacks a comprehensive overview. We begin by recognizing that answering any question about auctions must begin with an analysis of the equilibrium (or equilibria) of a game-theoretic model.[2] Without an equilibrium, any characterization of players' strategies or actions, and the final auction outcome, is unstable. Yet, characterizing an auction's equilibrium is important too because the characterization provides a prediction of players' behaviour.

Our survey provides an overview of the equilibria existence and characterization results in auctions, highlights the opportunities for research to economists and mathematicians and proceeds as follows. Section 2 describes the standard model of single-object auctions, and discusses the concepts of types, values, formats, as well as the suitability of the (Bayesian-)Nash equilibrium concept. (A reader familiar with game theory and basic auction theory can safely skip this section, but the information contained therein is assumed in later sections.) Section 3 presents general game theory results for discontinuous games and mixed strategy equilibria applicable to auction theory. Also, we explain why many standard existence results in general games do not apply to auctions, leading to the need for specific results. Section 4 details the common assumptions employed when proving equilibrium existence in pure strategies. Section 5 catalogues pure strategy equilibrium existence results for first-price auction across many different models, and we highlight open questions. Section 6 discusses

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how equilibria are characterized and surveys characterization results. Section 7 introduces multi-unit auctions, and surveys the equilibria and characterization results in standard multi-unit auctions, divisible good auctions and combinatorial auctions. Section 8 considers double auctions where sellers are active players. Section 9 connects auction theory to other branches of economics and briefly concludes.

2. Single-Object Auctions

The simplest auction setting has one (non-strategic) seller and many possible buyers competing for one indivisible object. Let n be the fixed number of bidders, where $i = 1, \ldots, n$ indexes each bidder.[3] In general, the object has an uncertain value to each participant and the values can be modelled as random variables, $V_1, \ldots, V_n$, one for each bidder. For risk neutral bidders, the payoff is 0 if the bidder loses the auction and does not receive the object, whereas the payoff is $V_i - p_i$ if the bidder wins the auction and receives the object.[4] The payment p_i for the object is determined by the bids in conjunction with the format of the auction.

The uncertainty about the payoff is in V_i, the value of the object. For example, if the object is a petroleum field, the value depends on the quantity and the quality of the oil in the field, the international price of petroleum, the cost of drilling and extraction and other private characteristics of bidder firms. In other words, many variables determine the value of the object. Among these variables, some are privately known by the bidders, some are known by other bidders and some variables are unknown to all participants. The information about those variables are captured by Harsanyi's theory of games with incomplete information through the concept of *types*.

2.1 *Types and Values*

In almost all cases, an auction is a game of incomplete information. It is an economic game because each player has to choose an action (bid) and the payoff to each player depends on the actions (bids) of the other bidders. Because of uncertainty regarding how much the object is worth to each player, the players do not know the payoffs even if all actions are given, and thus a game of incomplete information.

Harsanyi's remarkable papers made incomplete information games tractable.[5] Translating Harsanyi's theory of types to auctions, a bidder's value (V_i) is determined by the bidder's type (t_i), and potentially the types of all other bidders.[6] The list below introduces the standard concepts in auction theory regarding the connection between types and values:[7]

> *Private values*: The type of each bidder exclusively determines the value of the object for that bidder.[8] This setting is reasonable when the object for sale is exclusively used (consumed) by the buyer. Indeed, if the possibility of resale exists, then the types of the other bidders is potentially relevant for the value of the object in the resale.
>
> *Common values*: All bidders value the object equally.[9] This setting is natural for auctions of durable goods that can be resold.
>
> *Symmetric, interdependent values*: The value of the object depends on the bidder's type as well as the other bidders' types, although not on their identities.[10] This setting generalizes both the common and the private values settings.
>
> *Asymmetric, interdependent values*: No assumptions are made about the interaction between types and values for any of the bidders.

It is worth highlighting that common value is not the opposite of private value. Common value means that all bidders values the object equally, whereas private value means that the bidder possesses all information regarding the value of the object.

2.2 *Formats for Single-Object Auction*

The auction format determines the payment of the winning bidder. This survey only considers standard auctions in which the bidder that places the highest bid wins the object. The two main classes of standard auctions are sealed bid and open bid. In sealed formats, bidders submit their bids to the auctioneer without other bidders observing the bids. In the language of game theory, the bids are simultaneous actions.

Below we list the most common sealed bid auction formats:

First-price: the winner pays his bid.
Second-price: the winner pays the second highest bid.
All-pay: all bidders pay their bids.[11]
War-of-attrition: the winner pays the second highest bid and the losers pay their bids.[12]

Meanwhile, in open formats, the potential sale price changes during the auction as the auctioneer announces the current winning bid to all bidders, as bidding occur sequentially. Thus, open formats are sequential games. The different formats – sealed versus open – provide different information at different times to participants and therefore affect strategic actions.

For open auctions many possible formats also exist. A popular open auction is the English auction, where the bidders successively place higher bids to outbid opponents. When no bidder wants to place a higher bid, the auction ends and the declarer of the last bid wins the object and pays his bid.

A closely related open auction is the Japanese or button auction. In this auction, the price starts very low and each bidder presses a button, then the price increases continuously and bidders quit the auction (i.e. release the button) when the asking price for the object exceeds their willingness-to-pay for the object.[13] In open button auctions, all bidders see how many and (possibly) which bidders are still active. When there is just one bidder remaining, the auction finishes and last active bidder wins the object. The English and Japanese auctions are examples of ascending auctions because the price increases during the bidding.[14]

Milgrom and Weber (1982) and others usually model the Japanese auction when studying open ascending auctions because continuous price increases simplify the analysis. Indeed, formal models of the English auction are rare in the literature due to the complications from jump bidding where the price does not increase continuously; however, for examples of models with jump bids see Avery (1998) and Isaac *et al.* (2007).

Another example of an open auction is the Dutch auction. This auction begins by displaying a very high price for the object and then continuously lowers the price, until some bidder claims the object. The price is frozen when the object is claimed and the auction winner pays that price. Theoretically, the Dutch auction is equivalent to a first-price, sealed-bid auction, because no information is learned during the auction and the strategy is equivalent to deciding at what price to buy the object. Interestingly, empirical and experimental studies do not confirm this theoretical equivalence as in Kagel *et al.* (1987) and Lucking-Reiley (1999).

Similarly, the English and Japanese auctions are weakly equivalent to the second-price, sealed-bid auction. The weakness of the equivalence comes from the information that bidders learn with open formats. Specifically, each bidder learns the other bids and can update the value of the object for sale. Milgrom and Weber (1982) discuss the equivalence of the English and second-price auctions under the assumption of private values.

Both open and sealed-bid auctions can have reserve prices or entry fees. A reserve price means that the auctioneer will not sell the object for a price below the reserve price. Some auctions use a hidden reserve price. If the sale price is below the reserve price, then the object is not awarded to the winning bidder. In this situation, the object is said to be ‘bought in’, a terminology consistent with the notion that a reserve price acts like a bid from the seller. An entry fee is a fixed cost for the right to bid and in some circumstance increases seller revenues.[15]

For simplicity, this survey focuses on sealed-bid auctions. Thus, each bidder simply chooses a bid to maximize expected profit, where the difference between the value of the object and the payment determines profit when winning the auction, and zero otherwise.[16]

2.3 *Why Nash Equilibrium?*

One of the most commonly used concept in economics and game theory is the Nash equilibrium.[17] Despite the common usage, we find it important to discuss why the Nash equilibrium applies as an appropriate solution concept in the analysis of auctions.[18]

The main benefit of using the Nash equilibrium comes from its concise prediction of self-stable strategies by players. That is, no player can do better by unilaterally deviating from a Nash equilibrium strategy given that all other players continue to play their corresponding Nash equilibrium strategies. The stability of the equilibrium is an important requirement for an economist; otherwise, the derived characterizations and properties of the equilibrium would have little usefulness.

The Nash equilibrium solution concept, however, has some significant problems. An important criticism is that Nash equilibrium strategies are, in general, very difficult to calculate, especially in auctions where the number of potential bidder types and values, and potential bids, may be quite large. Compounding the complexity problem, if one player calculates an optimal strategy and is unsure that the other players will follow the corresponding strategies precisely, then the calculated equilibrium strategy may no longer be optimal.[19]

In response to the complexity critique, one can argue that players who do not learn the equilibrium strategy will be forced out of the market. In auctions, this means that experienced bidders, or the bidders that survived many auctions, learned to play the equilibrium strategies. Experimenters have confirmed this hypothesis in private values auctions (e.g. Kagel *et al.*, 1987). Nevertheless, the learning process interpretation has its own problems. It implicitly requires that the bidders participate in a number of similar auctions, with similar bidders, but preferably not the same. If the same bidders meet repeatedly in the auctions, then the likelihood of collusion increases. Also, the transitory period of learning the equilibrium may take a non-negligible amount of time. The learning period's length depends on the degree of the uncertainty and the information revealed at the end of the auction. Unfortunately, auction procedures often provide little information to participants. For example, the estimated values of the bidders are unobservable and, in some auctions, only the highest bid is known after the auction ends.[20]

Adding to the list of difficulties, some experimenters reported consistent bidding behaviour above the Nash equilibrium solution.[21] This leads to consideration of what can be said if the players do not follow Nash equilibrium strategies. As Pearce (1984) and Bernheim (1984) point out, rationality of the players does not imply Nash equilibrium, rather rationality only implies the use of rationalizable strategies. In this setting, Battigalli and Siniscalchi (2003) derive interesting conclusions and some of their results can explain experimental findings that contradict with predicted Nash equilibrium behaviour.

It is beyond the scope of this survey to present a complete picture of the debate regarding the relevance of Nash equilibrium. However, we highlight that, at some point, the Nash equilibrium solution concept needs to be 'assumed'. That is, the Nash equilibrium can be considered not only a solution concept, but additionally a meta-assumption for the methodology used by economists to analyse the behaviour of bidders in an auction. This meta-assumption, or paradigm, allows the economist to focus on behaviour that can be considered stable and reasonable, and from there begin to make predictions. Thus, the Nash equilibrium provides a powerful combination between the mathematical structure required to study complex games and the economic conclusions derived from the analysis.[22]

3. General Equilibria Results

Nash (1950) provides the first general existence result for equilibria in games.[23] The hypotheses of Nash's theorem are continuity and quasiconcavity of the payoff functions, and it guarantees existence in mixed strategies. In this section we address three issues. one, mixed strategy equilibria limit the ability to predict behaviour, mitigating the usefulness of the Nash equilibrium, and leading to a preference for pure strategy equilibria. two, auctions do not necessarily possess continuous payoff functions due to the positive probability of ties, and thus proving equilibria existence in auctions is generally more difficult compared to other economic games. Three, we discuss different methodologies the literature uses to handle the possibility of ties.

3.1 *Mixed Strategies*

When using a mixed strategy, the bidder chooses a probability distribution over the set of possible bids. Although mixed strategies are more general than pure strategies – because any pure strategy comes from a mixed strategy with a degenerate probability distribution that selects a particular bid with probability one – we highlight two problems interpreting mixed strategies in an auction setting.

First, it is difficult to accept that a bidder chooses their bid through a random device. Indeed, the reliance on a random device is a general objection to mixed strategies in game theory. In response, Rubinstein (1991) discusses three alternative interpretations of mixed strategies. One, the game can be viewed as the interaction between large populations where the games takes place after a random draw from the population, so the mixed strategy is viewed as the distribution of pure choices in the population. Two, the 'purification' idea posits that while the players' actions might appear random to the economic modeller, the actions are deterministic given exogenous information not captured by the model. Three, a mixed strategy could be the prior beliefs held by all other players regarding a player's possible actions.[24]

Secondly, the mixed strategy equilibrium is difficult to characterize because a mixed strategy does not provide a specific prediction of players' actions, but rather a portfolio of possible actions that may not all be realized. Therefore, an existence result in mixed strategies, in general, does not provide as much information about the bidding behaviour of the players, leading to a preference for existence results in pure strategies.

3.2 *Discontinuous Games*

An economic game is discontinuous when small changes in player behaviour can create large differences in the final payoff. Because discontinuous games are common, including in auction settings when ties may occur with positive probability, a vast literature exists studying this class of games. Contributors include Dasgupta and Maskin (1986), Simon (1987), Baye *et al.* (1993), Reny (1999), Carmona (2005), Bagh and Jofre (2006) and Carmona (2009).

The most general result that applies to auctions comes from Jackson *et al.* (2002). They guarantee the existence of an equilibrium in discontinuous games with all (atomless) distributions of types. Nevertheless, the result has at least two undesirable characteristics: first, it is in mixed strategy; and, secondly, it requires an endogenously determined tie-breaking rule. Jackson and Swinkels (2005) continues the analysis for a large class of private value games and shows that the outcome is invariant to the tie-breaking rule, yet the existence result still employs a mixed strategy.

3.3 *Tie-breaking Rules*

The payoffs in auctions are discontinuous when a positive probability of tied bids exists. In the case of a tie, a tie-breaking rule (or mechanism) must allocate an indivisible object. However, a bidder that wins the object with probability strictly less than one can bid above – but arbitrarily close to the tying bid – and in doing so discontinuously increases the probability of winning. Discontinuity in the payoff function makes the construction of optimal bidding strategies difficult to specify using the Nash equilibrium concept. The main question of this section becomes: What kind of tie-breaking rules are sufficient for equilibrium existence?

Lebrun (1996) seems to be the first to mention the need of special tie-breaking rules in the auction theory literature. Types of tie-breaking rules include:

> *Random device:* The standard way of solving ties appeals to a random device, like a coin, to decide the outcome of the game. Under this rule, some discontinuous games do not have equilibria.
> *Endogenous:* Simon and Zame (1990) point out that a special, endogenous tie-breaking rule can solve the problem of existence for general discontinuous games. Jackson *et al*. (2002) generalizes Simon and Zame (1990) for games with incomplete information. The endogeneity of the tie-breaking rule follows from the limit of allocations that may result from the auction. Using this approach, one cannot characterize the limit of allocations or describe a specific tie-breaking rule. However, Araujo and de Castro (2009) employ an endogenous tie-breaking rule that is monotonic in types, where the bidder with the higher type always wins the tie-breaking procedure.
> *Exogenous:* Maskin and Riley (2000b) adopt an exogenous tie-breaking rule for auctions with finite bidder types. The rule specifies that the bidders who tie participate in a second-price auction to determine the winner. Araujo *et al*. (2008) adopt an all-pay auction as the tie-breaking game, and they prove that this rule is sufficient to ensure equilibrium existence results in games where the utility function is not necessarily monotonic.

Again, for private values auctions, Jackson and Swinkels (2005) demonstrate that the equilibrium is invariant to the tie-breaking rule, diminishing the importance of rule specification. Unfortunately, the invariance result does not hold for interdependent values auctions, as illustrated in Jackson *et al*. (2002). Interestingly, Athey (2001) considers a general class of games with interdependent values, but her result does not require an endogenous tie-breaking rule. Araujo and de Castro (2009) discuss what assumption in Athey (2001) makes an endogenous tie-breaking rule unnecessary.

4. Standard Assumptions

Before surveying the pure strategy existence results for single-object auctions in Section 5, we introduce the usual assumptions that guarantee an equilibrium in monotonic bidding functions. A monotonic bidding function means that higher types lead to higher bids, all else equal, a strong characterization of bidder behaviour. Assumptions fall into three classes:

(i) Monotonicity of the value function (and The Single Crossing Property);
(ii) Statistical dependency of types; and,
(iii) Dimension of types and bids.

Many open question in auction theory involve relaxing these assumptions.

4.1 *Monotonicity of the Value Function*

The literature usually assumes that payoffs are strictly increasing in a bidder's own type and constant, or non-decreasing, in the other bidders' types. Then, monotonic methods ensure equilibrium existence by assuming a form of the Single Cross Property (SCP), such as in Athey (2001) or Reny and Zamir (2004). Although these papers work with non-differentiable versions of the SCP, the differentiable case SCP requires that the second cross derivative of types and bids are positive, and helps establish existence results.[25] The term ‘single crossing’ sometime refers to another condition that assures an efficient equilibrium, where the object is assigned to the player that values it the most.[26] It is important not to confound these two different concepts, since they are not logically related.

4.2 *Statistical Dependence of Types*

Next, most of the equilibrium results assume independence of types. Independence is a strong assumption and relaxing this assumption is quite valuable. Milgrom and Weber (1982) achieve remarkable success in relaxing independence through the concept of affiliation.[27] Under this kind of dependence, they prove the existence in first-price, second-price and English auctions. Since their paper, almost all models that relax the independence hypothesis in auction theory use the concept of affiliation. Jackson and Swinkels (2005) is a notable exception, and for a more recent discussion, see de Castro (2007). In another interesting model, Riley (1988) investigates a setting where information about the value of an object is correlated across bidders.

4.3 *Dimension of Types*

Under the last class of hypotheses, the literature usually assumes that types and bids are unidimensional, and can be simply expressed as real numbers. If one allows multidimensional types, then it is harder to determine which bidder values an object the most. Although generally restrictive, the unidimensionality of types does not restrict the case of private values with independent types. In this case, one can always reparameterize the multidimensional signals of each bidder into a unidimensional type given by

$$\tau_i \equiv E[V_i \mid t_i]$$

where, τ_i summarizes all the information that bidder i needs to know. In other words, each bidder can construct τ_i as a sufficient statistic.[28] In the case of dependent types, the reparameterizing argument becomes problematic, because the bidder does not want to lose information when summarizing information into a unidimensional variable.

In the case of interdependent values, the existence of sufficient statistics becomes less clear. Under a scenario of symmetric, interdependent values, but with independent types, Araujo *et al.* (2008) show that a sufficient statistic exists, assuming a strictly increasing value function. Unfortunately, they also show a loss of generality occurs when values are non-decreasing in types. It is still an open question whether dimension reduction can be done in the case of asymmetric auctions with non-symmetric, interdependent values or correlated types.

To be clear, a multi-dimensional bid can arise in single-object auctions. Multi-dimensionality can model situations such as the procurement of a service with many different characteristics. For instance, some companies and governments buy specialty goods and services through a mechanism where the potential suppliers need to specify not only prices, but also warranty, quality, time to delivery, and other characteristics. All these characteristics have to be taken in account to decide the winner in such instances. Che (1993) works with a model of multidimensional bids that captures the described situation.

5. Pure Strategy Equilibrium Existence

Here, we survey the pure strategy existence results for single-object auctions that began with the pioneering work of Vickrey (1961).[29] Roughly speaking, two methods are used in the literature for establishing an equilibrium in pure strategies. The first analyses the solution to a (set of) differential equation(s) derived from the first-order condition(s) of the bidder's best-reply problem, and then, under some assumptions, proves that this solution satisfies sufficient conditions for an equilibrium. The second method begins by restricting the feasible bids (or types) to discrete sets. With the auction reduced to a finite game, it is then proven to have an equilibrium. The properties of convergence guarantee the desired behaviour in the limit as the grid of actions (or types) becomes finer. The first method has the advantage of characterizing the equilibrium strategy and, in some cases, providing an analytical expression. On the other hand, the second method is more general.

Papers that use the first approach include Milgrom and Weber (1982), Lebrun (1999) and Lizzeri and Persico (2000). Milgrom and Weber (1982) assume affiliated types and that each bidders' utility function is increasing in all types. Under these conditions, they show that there exists a pure strategy equilibrium for which the bidding functions are monotonic in types. Their results cover the first-price, the second-price and English auctions but require, as an important condition, symmetry. Lebrun (1999) extends the differential equation approach to the asymmetric first-price auction, but with private values and independent types. Lizzeri and Persico (2000) use the approach for the case of common values with reserve price and affiliated types, but only two bidders.

Maskin and Riley (2000a), Athey (2001) and Reny and Zamir (2004) follow the second approach. Maskin and Riley (2000a) establish existence of equilibrium for asymmetric, first-price auctions either with affiliated types and private values, or with independent types and common values. The model in Athey (2001) covers a wider class of discontinuous games, establishing existence results for several two bidder first-price auctions with affiliated types and common values. Reny and Zamir (2004) extent Athey (2001) results to the n bidder case.

Table 1 summarizes the equilibrium existence results for first-price auctions under various assumptions about the statistical dependence of types and how types affect the value function. Assumptions about the value function run across the top of the table, whereas assumptions about types and the number of bidders appear on the left-hand side. We include the symmetric, private value assumption combination for completeness. The table also shows the cases that have no existence result denoted 'Open Question', where we define a question as open if the best known result is the existence of mixed strategy equilibria with endogenous tie-breaking rule in Jackson *et al.* (2002).

Assumption combinations for first-price auctions not present in Table 1 do not have existence results (unless otherwise noted). The question of existence remains open for many settings involving general type dependence and non-monotonic value functions. Section 6 discusses the characterizations presented in the table.

We continue with equilibrium results in other auction formats. Milgrom (1981) analyses the second-price auction. Milgrom and Weber (1982) provide an early analysis of the English auction assuming symmetric, private values with independent types, risk neutrality and unitary demand. Amann and Leininger (1996) and Krishna and Morgan (1997) study the cases of all-pay auctions and war-of-attrition.[30]

Almost all of the results in the papers about pure strategy equilibrium existence share a common characteristic: the equilibrium bidding functions are monotonic non-decreasing in the bidders' types. We note a few exceptions. In Zheng (2001), the private information is the budget constraint, and the bidding behaviour can be non-monotonic, but nevertheless his setting also has a monotonic equilibrium. McAdams (2007) provides examples of uniform price auctions (two units are sold) where non-monotonic equilibria can exist. Araujo *et al.* (2008) study auctions where the utility functions are not necessarily monotonic and non-monotonic equilibrium arise naturally. On the other hand, de Castro (2008) shows that symmetric first-price auctions with grid-distributions always have an equilibrium in pure strategies, but need not have monotonic strategies.[31]

Table 1. Summary of Existence and Characterization Results in First-Price Auctions.

	Value Function				
	Monotonic				Non-monotonic
	Symmetric		Asymmetric		Symmetric
Assumptions	Private	Interdependent	Private	Interdependent	Interdependent
Independent Types / 2-n bidders	Vickrey (1961), Griesmer *et al.* (1967)	Milgrom and Weber (1982)	Griesmer *et al.* (1967), Lebrun (1996, 1999)	Maskin and Riley (2000a)	Araujo *et al.* (2008)
Characterization:	(1)	(1)	(2)	(2)	(1)
Affiliated Types / 2 bidders	Milgrom and Weber (1982)	Milgrom and Weber (1982)	Milgrom and Weber (1982)	Lizzeri and Persico (2000), Athey (2001)	Open Question
Characterization:	(1)	(1)	(2)	(2)	
Affiliated Types / n bidders	Milgrom and Weber (1982)	Milgrom and Weber (1982)	Maskin and Riley (2000a)	Reny and Zamir (2004)	Open Question
Characterization:	(1)	(1)	(2)	(3)	
General Dependence/ n bidders	de Castro (2008)	Open Question	Open Question	Open Question	Open Question
Characterization:	(1), (4)[a]	(1)			

Notes: 'Open Question' means that the best known result is the existence of mixed strategy equilibria with endogenous tie-breaking rule Jackson *et al.* (2002). The characterization results are categorized as follows: (1) closed-form solution; (2) system of differential equation(s); (3) monotonicity of bidding functions in types; (4) existence in pure strategies. For more details of characterization descriptions and results, see Section 6.
[a]The closed-form solution in de Castro (2008) only holds for monotonic strategies.

6. Characterization

Characterizations play a key role in understanding the strategic behaviour of bidders. Some may even consider characterization results to be *more* important than existence results. A characterization provides a predicted behavioural action, or tendency, given bidders' types and therefore characterization results are particularly useful for auction practitioners. Generally speaking, characterizations allow comparative static exercises on such parameters as the number of bidders or the distribution of types.

We classify characterization results into four categories, organized from most specific to most general: closed-form solutions, systems of differential equations, monotonicity and pure strategies.[32] We describe the four categories and survey the relevant results as follows:

(i) *Closed-form:* In auctions, usually the first-order condition (FOC) leads to a differential equation, which sometimes has a closed-form solution.[33] This solution specifies an exact bid given a bidder's type (and possibly depending on some other parameters).[34] This type of explicit characterization is very convenient, but generally difficult to obtain. At present, closed-form results are limited to symmetric, single-object auctions and can be found in any textbook about auctions. Milgrom and Weber (1982) is an early paper with closed-form solutions to the first-price, second-price and English auctions; and, see Krishna and Morgan (1997) for the war-of attrition and all-pay auctions.

(ii) *System of differential equations:* When the FOCs do not yield a closed-form solution, they form a system of differential equations. Unfortunately, the system characterization does not provide as much information as the closed form solution. However, using differential equations one can study the tendencies of strategic behaviour as opposed to prescribing a specific bid. System characterization results apply to asymmetric, first-price auctions as in Lebrun (1998, 1999), Maskin and Riley (2000a) and Lebrun (2006). Williams (1991) derives system characterizations for double auctions. de Castro and Riascos (2009) generalize the first-order conditions for general auctions, including multi-unit auctions. However, we are not aware of any general result showing the existence of differentiable equilibria in multi-unit auctions.

(iii) *Monotonicity:* Without a system of equations, sometime one can determine that the bidding functions must exhibit monotonicity in types. Although an interesting result, this limited characterization contains less information than the general system of equations, but some auction settings limit the current literature to monotonicity characterizations. For instance, Athey (2001) and Reny and Zamir (2004), for single-object auctions, and McAdams (2003, 2006), for multi-object auctions.

(iv) *Pure strategy:* In addition to the three previous types of characterization, we can consider proving that an equilibrium exists in *pure* strategies, instead of mixed strategies, as a characterization result. Most of the literature derives pure strategy equilibria with monotonic strategies; however, non-monotonic pure strategy equilibria exist (e.g. de Castro, 2008).

In general, robust characterizations are scarce and thus many open questions about characterizations remain. Specifically, Table 1 shows the types of characterization for first-price auctions, where the number in each cell corresponds to the characterization type enumerated above. Clearly, increasing the specificity of any characterization provides value to researchers and practitioners.

7. Multi-Unit Auctions

Instead of just one object, the seller may have many objects to sell. In this case, the seller may select to implement a multi-unit auction where all units are sold in the same auction – in contrast to a sequence of single-object auctions. Multi-unit auctions also apply when a seller wants to sell perfectly divisible shares of an object.

All of the complexities in single-object auctions must also be addressed in the analysis of multi-unit auctions, but this expanded setting creates additional dimensions for strategic behaviour by buyers. Importantly, not all equilibrium results from single-object auctions have analogous results in multi-unit auctions. Open questions regarding equilibria in multi-unit auctions include completely describing the strategy space and discovering additional equilibrium existence results under relaxed assumptions.

7.1 *A Model of Multi-Unit Auction*

Let the seller have L fixed units for sale, so the value V_i becomes an L-vector. Often the elements of V_i exhibit diminishing marginal values. In a multi-unit auction each bidder submits a demand function; that is, a vector $b_i = (b_{i1}, \ldots, b_{iL})$, where b_{ih} specifies bidder i's willingness-to-pay for the hth unit that the bidder could receive. However, some multi-unit auction rules limit the number of steps in the demand function to an integer value below L. For a general model of multi-unit auctions with indivisible units, see de Castro and Riascos (2009).

7.2 *Formats of Multi-Unit Auctions*

As in the single-object case, the multi-unit auction has many possible formats. The most important sealed price formats for multi-unit auctions include:

Discriminatory ('pay-your-bid'): Each bidder pays exactly the bid for the unit received.
Uniform-price: All the units are sold at the same price.

- Highest loser: The price is set at the highest losing bid.
- Lowest winner: The price is set at the lowest winning bid.
- Convex combination: Auction rules may allow the price to be between the two previous options or the two previous options may occur each with some positive probability.[35]

Vickrey: For each unit bought, the seller pays only the minimum necessary to win such unit. That is, the winner of each unit pays the highest losing bid of the other bidders.

As in the case of single unit auctions, open formats exist in close correspondence with those given above. For details and a more lengthy treatment, see Krishna (2002).

7.3 *Existence Results for Multi-Unit Auctions*

Most results of equilibrium existence for multi-unit auctions assume homogeneous units for sale. Engelbrecht-Wiggans and Kahn (1998) obtain some of the first results specific to multi-unit auctions. The authors limit their analysis to two bidders with two objects for sale, but prove existence of an equilibrium for the highest-losing, uniform-price auction. Chakraborty (2006) follows by proving equilibrium existence in a discriminatory format. Alvarez and Mazon (2010) show that an equilibrium exists in a discriminatory auction for two units of common value.

Reny (1999) shows how his better-reply security condition implies the existence of equilibrium for multi-unit, private values, independent types, pay-your-bid auctions. Extending Athey (2001) to the multidimensional framework, McAdams (2003) proves the existence of pure strategy equilibrium in multi-unit, uniform price auctions.

All of the results presented in this section rely upon the assumption of monotone strategies where, all else equal, increasing a bidder's type raises the value of the object(s), the bid(s) and thus weakly increases likelihood of winning. However, the focus on monotone bidding strategies can be quite restrictive in some circumstances.

7.4 *Divisible Good Auction*

Another setting for the multi-unit auction occurs when a good is perfectly divisible. When selling shares of a good, the bidders may be able to submit a continuous demand function – instead of a step-function as required by the discrete case – specifying the willingness-to-pay for any fraction on the normalized interval [0, 1]. In an early analysis, Wilson (1979) studies a symmetric, common value, uniform price auction and compares the revenue from selling shares versus selling the entire unit in a single-object auction, and concludes that a share auction yields a significantly lower sale price. Kremer and Nyborg (2004) show the result in Wilson (1979) is due to a specific share allocation rule rationing excess demand. Wang and Zender (2002) prove the existence of a continuum of equilibria for bidders with symmetric information in both uniform and discriminatory auctions. Besides the uniform price per share auction, divisible good auctions can take other formats such as the discriminatory or Vickrey. For the general case when utility is not linear, equilibria existence is an open question when bidding occurs in continuous units and non-integer steps.

7.5 *Combinatorial Auctions*

A variant of the multi-unit auction is the combinatorial auction, where the goods are not required to be homogeneous. Instead, the heterogeneous objects are sold in packages so that bidders can exploit complementarities in the objects.[36] In doing so, the strategies for combinatorial auctions are extremely complex give the large number of possible packages and bids. The expansion of strategies can only increase the number of potential bidders and hopefully – from the seller's perspective – increase revenues.

An important feature of combinatorial auctions is the clearing mechanism that determines the price of each package. In this setting, prices can be non-linear and non-anonymous (see Bikhchandani and Mamer, 1997; Bikhchandani and Ostroy, 2002). Non-linear pricing means that the price paid for a package of two identical objects can cost more than the sum of the same two objects bought by different buyers. Non-anonymous pricing means that a seller can charge buyers different prices for the same packages.

Combinatorial auctions can be quite complex. Also, due to their generality, results regarding combinatorial auctions focus on mechanisms that support a desired equilibrium characteristic, rather than equilibrium existence. The Vickrey-Clarke-Groves (VCG) mechanism – a version of the second-price auction for multiple units – is a popular benchmark for new results, because the it elicits truthful bidding as the equilibrium strategy and it provides an efficient outcome.[37] In another combinatorial auction mechanism, Ausubel and Milgrom (2002) describe an open, ascending-price auction with jump bidding, designed to increase revenues to the seller. For more detailed surveys on combinatorial auctions, see de Vreis and Vorha (2003) and Cramton *et al.* (2006).

8. Double Auctions

In double auctions, both buyers act strategically. An example of a double auction is the standard financial market for equity shares, but *few* studies make the explicit connection between financial markets and double auctions. In fact, double auctions can be the correct model for many market institutions where players (bidders) can use their private information in order to manipulate the price of the object; however, as the number of players (bidders) increases, the power to manipulate price diminishes. This interesting line of investigation may establish connections between general equilibrium and strategic behaviour. Double auctions are also useful when analysing secret reserve prices, used in some auction houses, because secret reserve prices can be modelled as bids by sellers.

An important class of double auctions are the k-double auctions. These are sealed-bid auctions, where trade occurs if a buyer bid b is above a seller offer s. The k refers to a constant $k \in (0, 1)$

which determines the trading price $kb + (1 - k)s$ in case there is trade ($b \geqslant s$). That is, in a k-double auction the price paid by the buyer to the seller is a fixed convex combination (given by k) of the bid and the offer. In particular, if $k = 1$ ($k = 0$), the double auction is called a 'buyer's-bid auction' ('seller's-offer auction'). In a series of papers, Satterthwaite and Williams prove equilibrium existence and study the efficiency properties of the sealed-bid k-double auction.[38] In particular, Satterthwaite and Williams (1989b) discusses the rate of convergence to efficiency. These papers consider the symmetric value, independent types case.

More recently, Jackson and Swinkels (2005) considered multi-unit, asymmetrical, double auctions with a general distribution of types and private values. In this setting, they are able to prove that the tie-breaking rule does not matter, a result that does not hold in interdependent values. They also prove the non-triviality of equilibrium; that is, an equilibrium with positive probability of trade exists, assuming more than one buyer and one seller.[39]

Reny and Perry (2006) and Fudenberg *et al.* (2007) consider symmetric, double auctions with conditionally independent types and prove the existence of an equilibrium when the number of players is high. Araujo and de Castro (2009) prove the existence of non-trivial pure strategy equilibria in asymmetric, double auctions with interdependent values and a small number of players, but with an endogenous tie-breaking rule. Reny and Perry (2006) and Fudenberg *et al.* (2007) do not need special tie-breaking rules due to the number of players in their model. The assumption of large number of players ensures that the best replies are increasing, and is similar to the role that statistical independence plays in Araujo and de Castro (2009).

9. Connections and Conclusion

Beyond the traditional topics, the study of auctions has deep connections to other branches of economics. Some important questions in the general equilibrium theory can be rigorously addressed via auctions. For instance, Milgrom (1981) discusses how auctions can provide foundations for rational expectation theory. Reny and Perry (2006) contributes to this research program by extending Milgrom's idea to a model of double auctions. Auctions can also be used to build a foundation for the price-taking assumption that is central to general equilibrium. There is also a connection between Akerlof's 'market for lemons' and the winners' curse, as shown by Araujo and de Castro (2009). The impossibility theorem of Myerson and Satterthwaite (1983) generated a literature on efficient auctions such as Swinkels (2001) and Cripps and Swinkels (2006). Auctions are also convenient to address issues of information aggregation, a central question in economics, at least since Hayek (1945). For auctions, the explicit influence of types on bids shows how information available to individual buyers (and sellers) forms the final, market clearing price. For more recent works on information aggregation and auctions, see Kremer (2002). In sum, auctions are useful when addressing many important economic questions.

This survey offers an introduction to equilibria in auctions, reviews of the literature, and highlights open questions. Although some of the properties of auctions are well understood, many still require renewed efforts, and hopefully this survey helps to stimulate the necessary research.

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Notes

1. For textbooks, see Krishna (2002), Milgrom (2004), Klemperer (2004) and Menezes and Monteiro (2005). For surveys, see Milgrom (1989), Wolfstetter (1996), Klemperer (1999) and Maskin (2004).

2. Examples of important questions regarding auctions include: What auction format leads to the highest expected revenue for the seller? How do auction rules impact the number of participating bidders? How can the rules promote bidding? How does aggressive bidding by some participants affect other bidders' behaviour in equilibrium? How can collusion be prevented? What is the value of information? How does the seller use his information? Do the auction rules promote the revelation of information and, thus, good price formation? Is the auction efficient; that is, does it award the object to the bidder that values it the most? And so on.
3. Some papers analyse how the conditions and rules of the auction affect the number of participants where the number of participants is not assumed fixed (e.g. Levin and Smith, 1994). In complementary work, Jehiel and Moldovanu (1996) study the strategic incentives of potential bidders to participate in an auction.
4. There are some technical difficulties when modelling other attitudes towards risk. Also, a zero payment in the losing case assumes no cost for submitting a bid.
5. See Harsanyi (1967, 1968a,b). Appendix A provides a summary of Harsanyi's approach. For a mathematical formulation reconciling beliefs about other players' types, see Mertens and Zamir (1985).
6. Mathematically, $v_i(t_1, t_2, \ldots, t_n) = \mathbb{E}[V_i \mid t_1, t_2, \ldots, t_n]$, where v_i stands for the value function $v_i : T \to \mathbb{R}$, and $t_i \in T_i$ represents the Harsanyi type of player i. We usually write $t = (t_1, t_2, \ldots, t_n)$. See appendix A for details.
7. For a discussion regarding the statistical dependence of types, see Section 4.
8. In private values, $v_i(t)$ depends only on t_i.
9. Common values is mathematically defined by $v_i(t) = v(t)\forall i$. Some authors call this *pure* common value and use the term 'common value' for what we call interdependent value.
10. Mathematically, $v_i(t) = v(t_i, t_{-i})$, where $t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ and $v(t_i, t_{-i}) = v(t_i, t'_{-i})$ for all t'_{-i} permutations of the types in t_{-i}.
11. Although not very common in practice, the all-pay auctions can model, for instance, R&D races. In such a model, everybody pays the investment in research, but the firm that invests the most in R&D secures the patent first, and receives the payoff.
12. The war-of-attrition can model survival among a set of firms.
13. Here, 'button' need not be taken literally.
14. The Japanese auction has two variants. In one variant, the bidders do not see when others dropout and thus no information is learned during the auction. As a result, the 'blind' Japanese auction is equivalent to the second-price, sealed-bid auction.
15. In settings were bidders interact post-auction, Jehiel and Moldovanu (2000) discuss how reserve prices and entry fee can influence the degree of positive and negative externalities in the post-auction environment.
16. The analysis of open auctions presents complications because the open auction is dynamic. The sequential nature of the bids means that more information is revealed in the open auction the longer the auction proceeds. These complications lead to a general lack of equilibrium results for dynamic auctions. See Basar and Olsder (1982) for a discussion of dynamic non-cooperative game theory, and Vincent (1990) for dynamic auctions.
17. For readers unfamiliar with the Nash equilibrium and the Expected Payoff function, see Appendixes B and C, respectively for formal definitions. Because of the uncertainty in auction payoffs, the Nash equilibrium concept specializes to the Bayesian–Nash equilibrium. From now on, this survey refers to a Bayesian–Nash equilibrium simply as a Nash equilibrium.
18. Many different solution concepts have been proposed to analyse games, such as the quantal response equilibrium by McKelvey and Palfrey (1995), the S(1)-equilibrium by Osborne and Rubinstein (1998), the analogy-based equilibrium by Jehiel (2005) and cursed-equilibrium by Eyster and Rabin (2005). Battigalli and Siniscalchi (2003) analyse auctions from the point of view of rationalizability, instead of equilibrium.

19. Jehiel and Koessler (2006) recognize these limitations and develop a bounded rationality model that aggregates players actions over analogy classes. They solve these types of games using the analogy-based expectation equilibrium (ABEE).
20. Jehiel (2010) discusses the role of information disclosure as a mechanism to impact allocative efficiency and auction revenues.
21. See Kagel (1995) for a discussion and a survey of the corresponding literature.
22. Of course, this is a testable meta-assumption. Nevertheless, it is hard to test, even in experiments, where the conditions are controlled. Kagel (1995) describe an intense debate about the consistency between the observed behaviour of individuals in experiments and the assumption that they play Nash equilibria. Empirical tests provide further complications, as discussed by Laffont (1997).
23. Before Nash, the study of equilibrium focused on zero-sum games. Nash (1950) generalized the concept of an equilibrium and of equilibrium existence for non-zero-sum games. It is sometimes argued that Nash's concept is already present in Cournot's works. Thus, some authors call it the Cournot–Nash equilibrium.
24. Aumann and Brandenburger (1995) provide sufficient conditions such that a profile of conjectures leads to a Nash equilibrium.
25. Mathematically, the SCP is defined as $\partial^2_{bt} u \geqslant 0$.
26. This other condition is denoted by $\partial_{t_i} u_i \geq \partial_{t_j} u_j$. See Krishna (2002) and the references therein.
27. Affiliation can be defined as follows. Consider n real random variables $t_1, \ldots, t_n$ and let $f(\cdot)$ be its joint density function. Then, the variables are affiliated if $f(t) \cdot f(t') \leqslant f(t \vee t') \cdot f(t \wedge t')$, where $t \vee t' \equiv (\max\{t_1, t'_1\}, \ldots, \max\{t_n, t'_n\})$ and $t \wedge t' \equiv (\min\{t_1, t'_1\}, \ldots, \min\{t_n, t'_n\})$. Milgrom and Weber (1982) use definition without use of the density function.
28. It follows from this argument that the unidimensional signal can be interpreted as the value for the bidder, see Milgrom and Weber (1982, footnote 14, p. 1097). Also, see Milgrom (2004, p. 159–161) for a more detailed discussion.
29. Griesmer *et al*. (1967) provide other early contributions.
30. Even limiting the focus to papers that treat the equilibrium as a central question, it is impossible to cite every relevant paper. Other dimensions of complexity include risk aversion (see Matthews, 1983; Maskin and Riley, 1984), collusion (see McAfee and McMillan, 1992), entry of bidders (see Levin and Smith, 1994; Campbell, 1998), and financial constraints (see Zheng, 2001; Fang and Parreiras, 2002).
31. Grid distributions are distributions with a density function which is constant in squares covering the support of types. See de Castro (2008) for a formal definition and a discussion.
32. It would be possible to define another category of characterization: whether the equilibrium applies standard tie-breaking rules or not. Jackson and Swinkels (2005) show that auctions with private values do not need special tie-breaking rules, thus providing this kind of 'characterization'. However, their result applies to mixed strategies, whereas this section only discusses pure strategy results.
33. Appendix D shows how a differential equation can be obtained from the first-order condition in a first-price auction.
34. Note that the solution of a first-order condition is not always an equilibrium; generally, a second-order condition, or some analogous condition, must be satisfied. For a setting where the closed-form solution always exists, but frequently fails to be an equilibrium, see de Castro (2008).
35. If the price is chosen to be the lowest winning bid, then this is a multi-unit version of the first-price auction, but if it is the highest losing bid, then this is a multi-unit version of the second-price auction. In addition, the pay-your-bid auction is another multi-unit extension of the first-price auction, whereas the Vickrey auction is an extension of the second-price auctions.
36. For example, a telecommunications company may have a viable business if it is able to obtain two adjacent spectrum licenses, but not if only one is acquired. The combinatorial auction enables the company to avoid buying one license and then getting outbid for a second license.

37. However, the VCG has problems as highlighted by Ausubel and Milgrom (2006) and Rothkopf (2007). Some of these problems including the possibility of zero revenue, susceptibility to collusion and the occurrence of shill bidders.
38. See Satterthwaite and Williams (1989a), Satterthwaite and Williams (1989b) and Williams (1991).
39. A trivial, no-trade equilibrium always exists: buyers offer too little, sellers ask too much and nobody has an incentive to deviate.
40. The reader can consider the parameters as random variables correlated to the random variable V_i. In this case, the utility function in equation (1) is a conditional expectation.
41. This is valid if the parameters can be infinite dimensional. In this case, one of the parameters can refer to the function. The hypothesis that the parameter lies in an Euclidean space is the restrictive one.
42. The literature debates how well-founded the CPA is when modelling interactive decision problems, see Gul (1998) and Aumann (1998).

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Appendix A: Harsanyi Types and Value Function

Each bidder has some information about V_i, that leads to a conditional distribution of its value. This includes the case where the bidder knows with certainty the true value of V_i. Also, each bidder i forms beliefs about the values of V_j, for $j \neq i$ and about the beliefs that bidders $j \neq i$ have about V_i. Unfortunately, the logical loop continues because each bidder i has now to consider the beliefs of the bidders $j \neq i$ over the beliefs of i over the beliefs of j and so on.

This section replicates the construction of types by Harsanyi (1967, 1968a,b) with the objective to describe the value function. It is convenient and realistic to describe the value, V_i, as a function of a set of parameters, that is, to assume that

$$V_i = u_i\,(r_{0i}, r_{1i}, \ldots, r_{ni}) \tag{A1}$$

where r_{0i} is a vector of parameters that are unknown to all players and r_{ki} is a vector of parameters that are unknown to some of the players but are known to player k, for $k = 1, \ldots, n$.[40] The subscript i in the above parameters refers to the value of the object to player i. Without loss of generality, we can assume that the function u_i is common knowledge for all players.[41] Harsanyi assumes that the set of all possible values of the parameters r_{ki} is R_{ki}, (a subset of) an Euclidean space with the convenient dimension. The vector $r_k = (r_{k1}, \ldots, r_{kn})$, for $k = 1, \ldots, n$ describes the information that player k has about the functions u_i, for $i = 1, \ldots, n$. The vector $r_0 = (r_{01}, \ldots, r_{0n})$ summarizes the parameters that none players have about u_i. Of course, we can write V_i as function of the vector $r = (r_1, \ldots, r_n)$, where the unnecessary parameters do not influence the payoff. So, we can write

$$V_i = u_i\,(r_0, r) \tag{A2}$$

where r is also denoted by (r_i, r_{-i}), with $r_{-i} \equiv (r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n)$. The range of r is denoted by R and the range of r_{-i} by R_{-i}.

Harsanyi adheres to the Bayesian doctrine, that assumes that the players attribute subjective probabilities to the unknown parameters, that is, each player forms a subjective probability over R. Of course, we could again parameterize the possible probabilities and say that s_{ki} is the parameter that player k knows about the subjective probability of player i. Repeating the previous procedure, s_i denotes the vector of parameters known to bidder i. We end up by saying that each player forms a probability $\mu_i^{s_i}$, over the set of unknown parameters: $R_0 \times R_{-i} \times S_{-i}$. Now, it is assumed that $\mu_i^{s_i}$ is just a conditional probability from a prior distribution $\mu_i(\cdot)$ over $R_0 \times R \times S$, that is, it is assumed that $\mu_i^{s_i} = \mu_i\,(\cdot \mid s_i)$. Finally, it is assumed that the priors are equal across the players, that is, $\mu_1 = \cdots = \mu_n = \mu$. This is known as Common Prior Assumption (CPA). It implies that the differences between the subjective probabilities are due just to the differences in the parameters s_i.[42]

The parameters that are unknown to all players do not influence their behaviour in the auction. Thus, bidders can restrict attention to $\overline{V}_i$, the expected value of V_i with respect to r_0. That is, we can eliminate the vector r_0 in equation (A2), so that we have

$$\overline{V}_i = \overline{u}_i\,(r \mid s_i) = \int_{R_0} u_i\,(r_0, r)\,\mu\,(dr_0, r \mid s_i) \tag{A3}$$

Now, we embrace all the information that bidder i possesses before choosing the bid in a unique vector

$$t_i = (r_i, s_i) = (r_{i1}, \ldots, r_{in}, s_{i1}, \ldots, s_{in}) \tag{A4}$$

and call it the type (or the signal) of bidder i. It represents all information that he has about: his own payoff, the payoff of the others, his own beliefs and the beliefs of the others. Again we write $t = (t_1, \ldots, t_n)$ so that we can rewrite (A3) as

$$v_i\,(t) \equiv \overline{V}_i = \overline{u}_i\,(r \mid s_i)$$

The function $v_i : T \to \mathbb{R}$ is called the value function.

This model can generalize to other risk attitudes and other kind of games. Begin by assuming that the payoff u_i also depends on the actions of the players, denoted by vectors $a = (a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n \equiv A$. In this case, minor changes are needed in the construction above to obtain functions v_i which also depend on A and not only on T. Thus, in a generalized form, $v_i : T \times A \to \mathbb{R}$.

Appendix B: Definition of Nash Equilibrium

A Nash equilibrium is a profile of strategies such that, for each bidder, no strictly better strategy exists than the one specified in the profile *given* that the other bidders follow the strategies specified in the profile. That is, no bidder finds it optimal to unilaterally deviate from the profile strategy. The Nash equilibrium definition does not claim uniqueness of the profile. Note that an auction is a non-cooperative game.

Formally, the game is a triple (I, X, V), where I is the set of players, $X = \times_{i\in I} X_i$ is the set of actions or strategies (X_i is the set of actions available for player $i \in I$), and V is the payoff function $V : X \to \mathbb{R}^I$, where $V_i : X \to \mathbb{R}$ (the ith component of V) gives the payoff ($V_i(x)$) of player i for each profile of strategies $x = (x_i)_{i\in I} \in X$. We will adopt the standard (abuse) of notation of writing $x_{-i} \equiv (x_j)_{j\in I, j\neq i}$ and $x = (x_i, x_{-i})$. Now, we can define:

Definition of Nash equilibrium: A profile $x^* \in X$ is a Nash equilibrium of the game (I, X, V) if, for each $i \in I$, $V_i(x_i^*, x_{-i}^*) \geq V_i(x_i, x_{-i}^*)$, for all $x_i \in X_i$.

Appendix C: Definition of *Ex ante* and Interim Payoff Functions

Given the definition of types by Harsanyi, the concept of an expected payoff then makes sense for auctions. Recall that the final (*ex post*) payoff from an auction is a function of the difference between the value of the outcome – either getting the object or not – and the amount paid as a result of bidding on the object – in the all-pay auction every bidder always pays their bid. The expected (*ex ante*) payoff weights all the possible final outcomes by the probability of that payoff occurring through the Harsanyi types and strategic bids.

In some auction settings, the interim expected payoff is a more relevant concept. The interim period is when each bidder knows their type, but the types of the other bidders have not been revealed. In contrast, the *ex ante* period is when none of the bidders know their type and the *ex post* period is when all bidders know their type.

To clarify these concepts, it is necessary to introduce some notation. Let $T = \times_{i \in I} T_i$ be the space of Harsanyi's types, $A = \times_{i \in I} A_i$ be the set of actions available in the game of incomplete information and $v_i : T \times A \to \mathbb{R}$ be the payoff function for each player $i \in I$. A Bayesian game is a game (I, X, V) where $X_i = \{x : T_i \to A_i | x \text{measurable}\}$, and $V_i : X \to \mathbb{R}$, the *ex ante* payoff function, is given by

$$V_i\left(x_i, x_{-i}\right) = \int_T v_i\left(t_i, t_{-i}, x_i\left(t_i\right), x_{-i}\left(t_{-i}\right)\right) \mu\left(dt\right) \tag{C1}$$

where this integral exists. In this set-up, a Bayesian–Nash equilibrium is a Nash equilibrium of the game (I, X, V).

The interim payoff function is $\Pi_i : T_i \times A_i \times X_{-i} \to \mathbb{R}$, given by

$$\Pi_i\left(t_i, a_i, x_{-i}\right) = \int_T v_i\left(t_i, t_{-i}, a_i, x_{-i}\left(t_{-i}\right)\right) \mu\left(dt_{-i} \mid t_i\right) \tag{C2}$$

where the integration is done on t_{-i} with respect to the conditional probability given t_i, $\mu(dt_{-i} \mid t_i)$. Usually we omit x_{-i} in the notation above, if it is clear from the context; specially, if we are dealing with symmetric equilibria. We say that a_i^* is optimal for player i with type t_i against the strategies x_{-i}^* of i's opponents if for all $a_i \in A_i$:

$$\Pi_i\left(t_i, a_i^*, x_{-i}^*\right) \geqslant \Pi_i\left(t_i, a_i, x_{-i}^*\right) \tag{C3}$$

Harsanyi (1968a) shows that $x^* = (x_i^*)_{i \in I}$ is a Bayesian–Nash equilibrium of the game (I, X, V) if and only if for almost-all t_i, $a_i^* = x_i^*(t_i)$ satisfies (C3). That is, if $x_i^*(t_i)$ is optimal against the strategies x_{-i}^* of player i's opponents.

Appendix D: Differential Equation from a First-Order Condition

In this Appendix, we provide an example of how to obtain a differential equation from a first-order condition. Consider a single-object, first-price auction with symmetric, interdependent values and affiliated types for N bidders. Suppose that all other bidders $i \neq j$ follow the increasing and differentiable bidding strategy β. Let $G(\cdot \mid t_i)$ denote the conditional cumulative distribution function (CDF) of maximum of the other bidders' values given bidder i's type. Formally, $G(\cdot \mid t_i)$ is the conditional CDF of $Y_i \equiv \max_{i \neq j} t_j$, where t_j is the Harsanyi type of bidder j. Similarly, let $g(\cdot \mid t_i)$ be the associated condition density function of Y_i. The interim expected payoff to bidder i with type t_i who bids b

$$\Pi_i\,(t_i, b, \beta(\cdot)) = \int_0^{\beta^{-1}(b)} (v(t_i, t_{-i}) - b) g(t_{-i} \mid t_i)\, dt_{-i}$$
$$= \int_0^{\beta^{-1}(b)} v(t_i, t_{-i}) g(t_{-i} \mid t_i)\, dt_{-i} - bG(\beta^{-1}(b) \mid t_i)$$

To obtain the first-order condition, we take the derivative of Π_i with respect to b, which is

$$\left\{E[v(t_i, t_{-i}) \mid Y_i = \beta^{-1}(b), t_i] - b\right\} g(\beta^{-1}(b) \mid t_i) \frac{d\beta^{-1}(b)}{db} - G(\beta^{-1}(b) \mid t_i) \tag{D1}$$

Under a symmetric and increasing equilibrium, the optimal for bidder i with type t_i is to bid $b = \beta(t_i)$, that is, $\beta^{-1}(b) = t_i$. This implies that $\frac{d\beta^{-1}(b)}{db} = \frac{1}{\beta'(t_i)}$. Thus, making equation (D1) equal to zero and by rearranging terms we obtain the differential equation

$$\beta'(t_i) = \left\{E[v(t_i, t_{-i}) \mid Y_i = \beta^{-1}(b), t_i] - \beta(t_i)\right\} \frac{g(t_i \mid t_i)}{G(t_i \mid t_i)} \tag{D2}$$

Under some assumptions (affiliation, for instance), it is possible to show that the solution to this differential equation is an equilibrium.