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DYNAMIC QUANTILE MODELS OF RATIONAL BEHAVIOR

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This paper develops a dynamic model of rational behavior under uncertainty, in which the agent maximizes the stream of future τ-quantile utilities, for $\tau \in (0, 1)$. That is, the agent has a quantile utility preference instead of the standard expected utility. Quantile preferences have useful advantages, including the ability to capture heterogeneity and allowing the separation between risk aversion and elasticity of intertemporal substitution. Although quantiles do not share some of the helpful properties of expectations, such as linearity and the law of iterated expectations, we are able to establish all the standard results in dynamic models. Namely, we show that the quantile preferences are dynamically consistent, the corresponding dynamic problem yields a value function, via a fixed point argument, this value function is concave and differentiable, and the principle of optimality holds. Additionally, we derive the corresponding Euler equation, which is well suited for using well-known quantile regression methods for estimating and testing the economic model. In this way, the parameters of the model can be interpreted as structural objects. Therefore, the proposed methods provide microeconomic foundations for quantile regression methods. To illustrate the developments, we construct an intertemporal consumption model and estimate the discount factor and elasticity of intertemporal substitution parameters across the quantiles. The results provide evidence of heterogeneity in these parameters.

KEYWORDS: Quantile utility, dynamic programing, quantile regression, intertemporal consumption.

1. INTRODUCTION

MODELING DYNAMIC ECONOMIC BEHAVIOR has been a concern in economics for a long time (see, e.g., Samuelson (1958), Baumol (1959), Koopmans (1960), Brock and Mirman (1972)). These models are critical for learning about economic effects, incentives, and to design policy analysis. We contribute to this literature by developing a new dynamic model for an individual, who, when selecting among uncertain alternatives, chooses the one with the highest τ-quantile of the stream of future utilities for a fixed $\tau \in (0, 1)$, instead of the standard expected utility. This quantile preference model is tractable, simple to interpret, and substantially broadens the scope of economic applications, because it allows to ac-

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count for heterogeneity through the quantiles and separation between risk aversion and elasticity of intertemporal substitution.[1]

Quantile preferences were first studied by Manski (1988) and axiomatized by Chambers (2009) and Rostek (2010). Manski (1988) developed the decision-theoretic attributes of quantile maximization and examined risk preferences of quantile maximizers. In the context of preferences over distributions, Chambers (2009) showed that monotonicity, ordinal covariance, and continuity characterize quantile preferences. Rostek (2010) axiomatized the quantile preferences in Savage's (1954) framework, using a 'typical' consequence scenario.

This paper initiates the use of quantile preferences in a dynamic economic setting by providing a comprehensive analysis of a dynamic rational quantile model. To motivate our model, we begin by describing a setting where the dynamic quantile preferences are compelling.[2] The quantile preferences are represented by an additively separable quantile utility model with standard discounting. The associated recursive equation is characterized by the sum of the current period utility function and the discounted value of the certainty equivalent, which is a quantile function. In addition, we discuss the notion of risk attitude and elasticity of intertemporal substitution (EIS) in our model, and show that by using the recursive quantile preferences, it is possible to separate the notion of risk attitude from the intertemporal substitution. Thus, quantile preferences are a useful alternative to the expected utility, and a plausible complement to the study of rational behavior under uncertainty.[3]

We then introduce the dynamic programming for intertemporal decisions whereby the economic agent maximizes the present discounted value of the stream of future τ-quantile utilities by choosing a decision variable in a feasible set. Our first main result establishes dynamic consistency of the quantile preferences, in the sense commonly adopted in decision theory. Second, we show that the optimization problem leads to a contraction, which therefore has a unique fixed point. This fixed point is the value function of the problem and satisfies the Bellman equation. Third, we prove that the value function is concave and differentiable, thus establishing the quantile analog of the envelope theorem. Fourth, we show that the principle of optimality holds. Fifth, using these results, we derive the corresponding Euler equation for the infinite horizon problem. To obtain our Euler equation, we offer a sufficient condition for exchanging derivative and quantile operators, which does not hold in general.

We note that the theoretical developments and derivations in this paper are of independent interest. The main results for the dynamic quantile model—dynamic consistency, value function, principle of optimality, and Euler equation—are parallel to those of the expected utility model. However, because quantiles do not share all of the convenient properties of expectations, such as linearity and the law of iterated expectations, the generalizations of the results from expected utility to quantile preferences are not straightforward.

The derivation of the Euler equation is an important feature of this paper because it allows to connect economic theory with empirical applications. We show that the Euler equation has a conditional quantile representation and relates to quantile regression

[1]Rostek (2010) discussed several advantages of the static quantile preference, such as robustness, ability to deal with categorical (instead of continuous) variables, and the flexibility of offering a family of preferences indexed by quantiles.

[2]A formal axiomatization of these preferences was provided by de Castro and Galvao (2018).

[3]Quantile preferences can be associated with Choquet expected utility (see, e.g., Chambers (2007), Bassett, Koenker, and Kordas (2004)). The method of Value-at-Risk, which is widespread in finance, also is an instance of quantiles (see, e.g., Engle and Manganelli (2004)).

econometric methods, and hence, our methods provide microeconomic foundations for quantile regression. The Euler equation, which must be satisfied in equilibrium, implies a set of population orthogonality conditions that depend, in a nonlinear way, on variables observed by an econometrician and on unknown parameters characterizing the preferences. Thus, empirically, one can employ practical existing econometric methods, such as instrumental variables for nonlinear quantile regression, for estimating and testing the parameters of the model. In this fashion, these parameters can be interpreted as structural objects. In addition, varying the quantiles τ enables one to empirically estimate a set of parameters of interest as a function of the quantiles.[4]

Finally, we briefly illustrate the methods with an intertemporal consumption model, which is central to contemporary economics and finance. We use a variation of Lucas's (1978) model where the economic agent decides on how much to consume and save by maximizing a quantile utility function subject to a linear budget constraint. We solve the dynamic problem and obtain the Euler equation. Following a large body of literature, we specify an isoelastic utility function and estimate the implied discount factor and EIS parameters at different levels of risk attitude (quantiles). The empirical results document evidence that both parameters vary across quantiles.

More broadly, this paper contributes to the literature by proposing methods that could be applied to any dynamic economic problem, substituting the standard expected utility by dynamic quantile preferences. Such preferences maintain useful characteristics of the standard model, such as dynamic consistency and monotonicity. Moreover, they allow the separation between risk aversion and EIS. Since dynamic economic models are now routinely used in many fields, such as macroeconomics, finance, international economics, public economics, industrial organization, labor economics, and scenario-based analysis, among others, our methods expand the scope of economic analysis and empirical applications, providing an alternative to expected utility models.

The remainder of the paper is organized as follows. Section 2 presents definitions and basic properties of quantiles, provides a motivating example of recursive quantile preferences, and briefly discusses its properties. Section 3 describes the dynamic economic model and presents the main theoretical results. Section 4 illustrates the empirical usefulness of the new approach by applying it to an intertemporal consumption model. Finally, Section 5 concludes. We relegate all proofs to the Appendix.

1.1. *Review of the Literature*

This paper has a broad scope and relates to a number of streams of literature in economic theory and econometrics.

First, the paper relates to the extensive literature on dynamic nonlinear rational expectations models. Many models of dynamic maximization that use expected utility have been proposed and discussed. These models have been workhorses in several economic fields. We refer the reader to more comprehensive works, such as Stokey, Lucas, and Prescott (1989) and Ljungqvist and Sargent (2012). Another related segment of the literature studies recursive utilities. We refer the reader to Epstein and Zin (1989), Marinacci and Montrucchio (2010), Bommier, Kochov, and Le Grand (2017), among others. We extend this literature by replacing expected utility and its variations with quantile utility.

[4]We note that the theoretical methods do not impose restrictions across quantiles, and thus the parameter estimates might (or might not) vary across quantiles.

Second, this paper is related to a few works on economic models using the quantile preferences, such as Manski (1988), Chambers (2007, 2009), Bhattacharya (2009), Rostek (2010), and, especially, Giovannetti (2013). The latter studies a two-period economy for an intertemporal consumption model under quantile utility maximization. We contribute to this line of research by taking the quantile maximization to a general dynamic optimization model and deriving its properties.

Third, the paper relates to an extensive literature on estimating Euler equations. Since the contributions of Hall (1978), Lucas (1978), Hansen and Singleton (1982), and Dunn and Singleton (1986), it has become standard in economics to estimate Euler equations based on conditional expectation models. There are large bodies of literature in micro- and macroeconomics on this subject. We refer the reader to Attanasio and Low (2004) and Hall (2005), and the references therein, for a brief overview. The methods in this paper derive a Euler equation that has a conditional quantile function representation and estimate it using existing quantile regression (QR) econometric methods.

Finally, this paper relates to the QR literature, for which there is a large body of work in econometrics.[5] In a seminal paper, Koenker and Bassett (1978) introduced QR methods for estimation of conditional quantile functions. These models have provided a valuable tool in economics and statistics applications to capture heterogeneous effects, and for robust inference when the presence of outliers is an issue (see, e.g., Koenker (2005)). QR has been largely used in program evaluation studies (Chernozhukov and Hansen (2005) and Firpo (2007)), identification of nonseparable models (Chesher (2003) and Imbens and Newey (2009)), nonparametric identification and estimation of nonadditive random functions (Matzkin (2003)), and testing models with multiple equilibria (Echenique and Komunjer (2009)). This paper contributes to the effort of providing microeconomic foundations for QR by developing a dynamic optimization decision model that generates a conditional quantile restriction (Euler equation).

2. QUANTILE PREFERENCES

This section provides motivations for the recursive equation (equation (10) below) that is characterized by the sum of the current period utility function, and the discounted value of the quantile certainty equivalent. The recursive equation is the central element in the definition of the dynamic quantile preferences, which is completed in Section 3.

2.1. *Preliminaries*

Let $F_X : \mathbb{R} \to [0, 1]$ be the continuous and strictly increasing cumulative distribution function (c.d.f.) of the random variable X. Then, F_X has an inverse and we can define $\mathrm{Q}_\tau[X] \equiv F_X^{-1}(\tau)$, for $\tau \in (0, 1)$.[6] This definition can be extended to the conditional case: if $F_{X|Y=y}(\cdot)$ is the conditional c.d.f. of X given $Y = y$, $\mathrm{Q}_\tau[X|Y = y] \equiv F^{-1}_{X|Y=y}(\tau)$. In general, we will write only $\mathrm{Q}_\tau[X|Y]$ for the corresponding random variable, as usual. In the Appendix, we give the definition of $\mathrm{Q}_\tau[X]$ for general c.d.f.'s and develop some useful properties of quantiles, such as the fact that it is left-continuous and $F_X(\mathrm{Q}_\tau[X]) \geq \tau$. Another well-known and useful property of quantiles is "invariance" with respect to monotonic transformations, that is, if $g : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function,

[5]This paper is also related to an econometrics literature on identification, estimation, and inference of general conditional moment restriction models. We refer the reader to, among others, Newey and McFadden (1994), Chen, Linton, and van Keilegom (2003), and Chen, Chernozhukov, Lee, and Newey (2014).

[6]For convenience, throughout the paper we will focus on $\tau \in (0, 1)$, unless explicitly stated.

then[7]

$$Q_\tau\big[g(X)\big] = g\big(Q_\tau[X]\big). \tag{1}$$

Note that this property is valid for an expectation operator only if g is affine, that is, $g(x) = ax + b$, for $a \geq 0$, or the distribution is a point mass. Hence, expectation is invariant with respect to positive affine (cardinal) transformations, and quantile is invariant with respect to monotonic (ordinal) transformations. This observation suggests the convenience of contrasting other properties of quantiles and expectations.

Both quantiles and expectations are monotonic in the following sense: if X first-order stochastically dominates Y, then $Q_\tau[X] \geq Q_\tau[Y]$. If X is risk-free, that is, $X = x$ with probability 1 for some x, then $Q_\tau[X] = x = E[X]$. Both are also translation-invariant, that is, $Q_\tau[\alpha + X] = \alpha + Q_\tau[X]$, $\forall\alpha \in \mathbb{R}$; and scale-invariant, that is, $Q_\tau[\alpha X] = \alpha Q_\tau[X]$, $\forall\alpha \in \mathbb{R}_+$. Notice, however, that $E[\alpha X] = \alpha E[X]$, $\forall\alpha \in \mathbb{R}$, while this equality is not true for quantiles with $\alpha < 0$. Indeed, $Q_\tau[-X] = -Q_{1-\tau}[X]$. On the other hand, quantiles do not share many of the convenient properties of expectations. We highlight three properties that fail for quantiles and would be important for our results. First, in general, quantiles are not linear: $Q_\tau[X + Y] \neq Q_\tau[X] + Q_\tau[Y]$, although Proposition A.4 in the Appendix provides a comonotonicity condition under which this additivity holds. Second, quantiles do not satisfy an analogue of the law of iterated expectations: if $\Sigma_0 \subset \Sigma_1$ are two σ-algebras, then, in general, $Q_\tau[Q_\tau[X|\Sigma_1]|\Sigma_0] \neq Q_\tau[X|\Sigma_0]$; see Example 3.7. Third, it is not possible to interchange a differentiation and a quantile operator, as it is for expectations. That is, in general, $\frac{\partial Q_\tau}{\partial x}[h(x, Z)] \neq Q_\tau[\frac{\partial h}{\partial x}(x, Z)]$. Proposition 3.19 below, to the best of our knowledge, is another original contribution of this paper and establishes sufficient conditions for this interchangeability.

2.2. *Static Quantile Preferences*

The quantile preferences over random variables X and Y can be defined as follows:

$$X \succcurlyeq Y \quad \Longleftrightarrow \quad Q_\tau[X] \geq Q_\tau[Y]. \tag{2}$$

It is useful to compare the quantile preference (2) with the standard expected utility preference, defined by

$$X \succcurlyeq Y \quad \Longleftrightarrow \quad E\big[u(X)\big] \geq E\big[u(Y)\big], \tag{3}$$

for some $u : \mathbb{R} \to \mathbb{R}$. First, notice that a utility function u appears in (3) but not in (2). The reason for such omission is the fact that we can use (1) to conclude that

$$\begin{aligned} X \succcurlyeq Y \quad &\Longleftrightarrow \quad Q_\tau[X] \geq Q_\tau[Y] \quad \Longleftrightarrow \quad u\big(Q_\tau[X]\big) \geq u\big(Q_\tau[Y]\big) \\ &\Longleftrightarrow \quad Q_\tau\big[u(X)\big] \geq Q_\tau\big[u(Y)\big], \end{aligned} \tag{4}$$

for any strictly increasing and continuous u. Thus, u does not play any role in the definition of the preference. In other words, (2) is invariant to any monotonic transformation of u, while the expected utility preference (3) is invariant only to monotonic affine transformations of u.

An important implication of (4) is that, while the concavity of u implies risk aversion for an expected utility maximizer, it does not have any impact on the risk attitude of a

[7]In fact, (1) is also valid for a left-continuous and non-decreasing function; see Lemma A.2 in the Appendix.

quantile maximizer: any concave or convex increasing u represents the same preference. Manski (1988) and Rostek (2010) argued that the risk attitude of a quantile maximizer can be captured by τ. We discuss this characterization in Section 2.4.2 below.

Manski (1988) was the first to study quantile preferences, which was recently axiomatized by Chambers (2009) and Rostek (2010). Rostek (2010) axiomatized the quantile preferences in the context of Savage's (1954) subjective framework. Rostek (2010) modified Savage's axioms to show that they are equivalent to the existence of a $\tau \in (0, 1)$, a probability measure, and a quantile utility function.[8,9] Chambers (2009) worked in a risk setting where the probability distribution of the random variables is given, in contrast with Rostek's uncertainty setting. He showed that the preference satisfies monotonicity, ordinal covariance, and continuity if and only if the preference is a quantile preference.[10]

2.3. *Recursive Quantile Preferences*

This section aims to motivate the recursive quantile preferences. First, we present an example where the dynamic quantile preference is particularly compelling. Second, we briefly discuss an axiomatization of the preferences. This axiomatization shows an important property of the recursive quantile preferences, which is the separation between risk attitudes and attitudes toward intertemporal substitution.

2.3.1. *Example*

This example is developed around three main ingredients: (i) a setting where a decision maker's preference is a quantile preference; (ii) a discussion of time aggregation in the deterministic case through the standard additive time aggregation preferences; (iii) a requirement of dynamic consistency for the choices made in different periods.

Assume that a decision maker (DM) chooses one among different policies that will affect her utility derived from consumption during T periods. The utility of the DM is expressed by $U = U(c_0, c_1, \dots, c_T)$, where c_t is the consumption in period $t = 0, 1, \dots, T$. Let $z_1, \dots, z_t$ be a sequence of random shocks. Each z_t is realized and known at the beginning of period t. Let $z^t = (z_1, \dots, z_t)$ represent the history of shocks up to period t, upon which the consumption in period t may depend, that is, $c_t = c_t(z^t)$ and c_0 is certain. The policy choice affects the distribution of the c_t and, hence, of U; see details in Section 4. The DM is particularly concerned about the τ-quantile of the distribution of the utility of consumption. For example, if $\tau = 0.1$, the DM is interested in the lower part of the distribution, while if $\tau = 0.99$, the interest is on the top 1% of the distribution. Thus, she maximizes

$$\mathrm{Q}_\tau[U] = \mathrm{Q}_\tau\big[U(c_0, c_1, \dots, c_T)\big] = \mathrm{Q}_\tau\big[U\big(c_0, c_1(z_1), \dots, c_t\big(z^t\big), \dots, c_T\big(z^T\big)\big)\big]. \tag{5}$$

Notice that the quantile τ captures the risk attitude of the DM, as briefly mentioned before and further discussed in Section 2.4.2 below. Hence, when the DM with risk attitude

[8] If $\tau \in \{0, 1\}$, the statement is more complex; see her paper for details.

[9] Rostek (2010) also showed that the quantiles preferences are probabilistic sophisticated for $\tau \in (0, 1)$, by using a variation of the original concept of probabilistic sophistication introduced by Machina and Schmeidler (1992).

[10] Since the upper semicontinuity property is a technical condition and first-order stochastic dominance is a mild restriction, also satisfied by expected utility, the really important property is invariance with respect to monotonic transformations, which is summarized by (1).

τ chooses a certain policy, she ensures to receive a utility level that is larger than $\mathrm{Q}_\tau[U]$ with probability $1-\tau$.

The setup (5) is related to a large literature on Value-at-Risk (VaR); see, for instance, Duffie and Pan (1997) and Gourieroux and Jasiak (2010). While VaR is usually used as a restriction of the domain of choice in the usual mean-maximization program, Gaivoronski and Pflug (2005, p. 4) discussed a model for portfolio optimization for short observation periods where the investor or manager decides on the optimal portfolio allocation to maximize the τ-quantile of the return of the portfolio at the end of the period (the wealth), since the mean is approximately zero for any reasonable portfolio. In a VaR setting, τ is referred to as confidence level and captures risk tolerance.

Having the general form of the objective function $\mathrm{Q}_\tau[U(c_0, c_1, \dots, c_T)]$, we now discuss the particular specification of $U(c_0, c_1, \dots, c_T)$, which includes the time aggregation across periods. Let us first fix attention to the riskless case. Assume that the DM faces a sequence of certain consumption levels $c_0, c_1, \dots, c_T$. The aggregation of this stream may take many forms, but the most common in economics is the standard exponential discounting:

$$U(c_0, c_1, \dots, c_T) = \sum_{t=0}^{T} \beta^t u(c_t), \tag{6}$$

for some $\beta \in (0, 1)$ and strictly increasing and continuous u. Notice that the preference on riskless streams of consumption defined by (6) is uniquely determined by u up to *affine* transformations. In particular, the preference will no longer be invariant with respect to all monotonic transformations of u as it is in the static case (3).[11] This is an important observation and its consequences will be further discussed below. It should be noted that it is possible to define alternative forms of the time aggregation that are invariant with respect to all monotonic transformations. An example is $U(c_0, c_1, \dots, c_T) = \min\{u(c_1), \dots, u(c_T)\}$. However, our motivation of quantile preferences does not rely on this invariance, while it is a central piece of Chambers's (2007) justification for the quantile function. In fact, invariance to all monotonic transformations *does not* hold in our model. We forgo this strong form of invariance to maintain the standard discounting for deterministic prospects.

Now we treat the case with risk. If there is no decision to be made in different periods, the introduction of risk poses no problem and the objective function could be written as $\mathrm{Q}_\tau[U] = \mathrm{Q}_\tau[\sum_{t=1}^{T} \beta^t u(c_t)]$.[12] However, if there is a decision to be made in each period, which is the case in most models, then we need to be more explicit about what is known at that time period. Moreover, it is important to require the choices to be dynamically consistent. Intuitively, the DM correctly anticipates future decisions such that there is no preference change over time.[13] It turns out that dynamic consistency is the last requirement that allows us to pin down the dynamic quantile preference.

[11]To see this, it is sufficient to observe that, for instance, $u(c_1) + \beta u(c_2)$ and $\sqrt{u(c_1)} + \beta\sqrt{u(c_2)}$ represent different orders in general.

[12]Notice that this model retains the standard additive separability in time, but not in uncertainty. This is in contrast with the dynamic expected utility model. Additive separability in time but not in uncertainty also appears in Epstein and Schneider (2003), Maccheroni, Marinacci, and Rustichini (2006), and Klibanoff, Marinacci, and Mukerji (2009), among others. As Mongin and Pivato (2015) discussed in more detail, full additive separability is implied by monotonicity axioms applied to both time and uncertainty.

[13]A formal definition of dynamic consistency is given in Section 3.4.

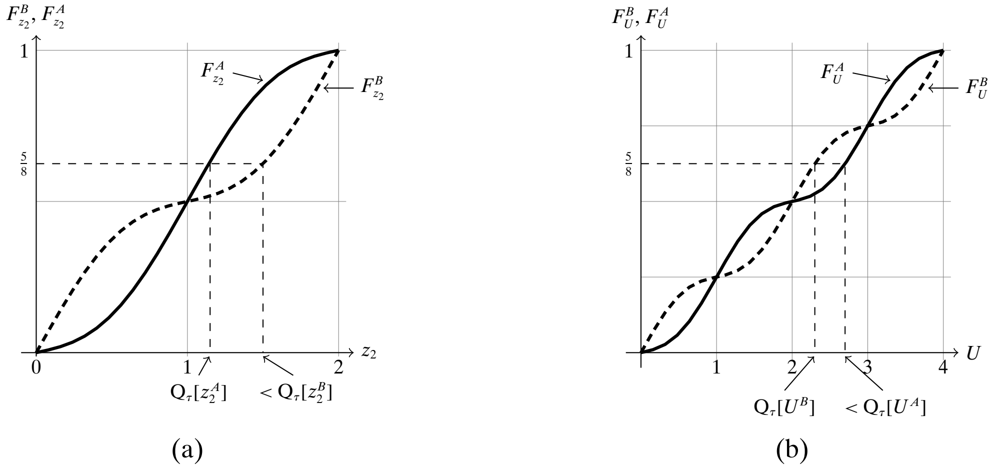


FIGURE 1.—Graph of the c.d.f. of the different variables, with respective quantiles.

In order to further discuss dynamic consistency, let us particularize the above setting. Let $T = 2$ and consider independent variables z_1, z_2, where z_t is revealed at period $t \in \{1, 2\}$. Let $U(c_0, c_1, c_2) = U(c_0, c_1(z_1), c_2(z_1, z_2)) = z_1 + z_2$ be the utility function. The DM has to choose an economic policy that determines the distribution of z_2, leaving the distribution of z_1 unaffected. Indeed, z_1 assumes values 0 or 2 with equal probability, no matter what is the policy. There are two policies, which for simplicity we call policies A and B. Let us denote by z^A_2 the random variable z_2 when policy A is chosen, and similarly for z^B_2. The c.d.f. of each of those variables is shown in Figure 1(a).[14] Similarly, we call U^A the utility when the policy A is adopted and U^B when B is, that is, $U^A = z_1 + z^A_2$ and $U^B = z_1 + z^B_2$. This highlights that the utility has different distributions, according to the policy chosen. The c.d.f.'s of U^A and U^B are depicted in Figure 1(b).[15]

Let us assume that the DM chooses her plan at the initial time (before z_1 is realized), but the actual choice can be made after the first period. The DM prefers A if

$$\mathrm{Q}_\tau\big[U^A\big] = \mathrm{Q}_\tau\big[z_1 + z^A_2\big] \geq \mathrm{Q}_\tau\big[z_1 + z^B_2\big] = \mathrm{Q}_\tau\big[U^B\big], \tag{7}$$

and B if the opposite is true. If, for instance, $\tau = \frac{5}{8}$, then $\mathrm{Q}_\tau[U^A] > \mathrm{Q}_\tau[U^B]$, that is, before z_1 is realized, the DM chooses A; see Figure 1(b). We will now see that, after the realization of z_1, this choice is inconsistent no matter what the actual value of z_1 is. Indeed, if $z_1 = 0$, then $z_1 + z_2 = z_2$ and $\mathrm{Q}_\tau[z^A_2] < \mathrm{Q}_\tau[z^B_2]$; see Figure 1(a). Similarly, if $z_1 = 2$, $z_1 + z_2 = 2 + z_2$ and $\mathrm{Q}_\tau[2 + z^A_2] = 2 + \mathrm{Q}_\tau[z^A_2] < 2 + \mathrm{Q}_\tau[z^B_2] = \mathrm{Q}_\tau[2 + z^B_2]$. In other words, in the moment that the choice is actually made, the DM chooses B, for all realizations of z_1. Thus, her choice before z_1 is realized should also be B and not A as implied by the model in (7).

This discussion shows that the model in (7) does not satisfy dynamic consistency. In order to obtain a model with such a property, we need to build a recursive structure in the objective function. Namely, we first evaluate the conditional quantile of z_2 given z_1.

[14]For instance, we consider that z_2's distribution is $F^A_{z_2}(x) = \frac{1}{2}[x - \frac{1}{4}\sin(\pi x)]$ if the A policy is adopted, while z_2's distribution is $F^B_{z_2}(x) = \frac{1}{2}[x + \frac{1}{4}\sin(\pi x)]$ if the policy is B, for $x \in [0, 2]$.

[15]U^A's c.d.f. is $F^A_U(x) = \frac{1}{4}[x - \frac{1}{4}\sin(\pi x)]$ and U^B's c.d.f. is $F^B_U(x) = \frac{1}{4}[x + \frac{1}{4}\sin(\pi x)]$, for $x \in [0, 4]$.

In this particular example, where the variables are independent, $Q_\tau[z_2|z_1] = Q_\tau[z_2]$, but in the general case, we will need to consider the random variable $g(z_1) = Q_\tau[z_2|z_1]$. Then, we evaluate the quantile of $z_1 + g(z_1)$, that is,

$$Q_\tau\big[z_1 + g(z_1)\big] = Q_\tau\big[z_1 + Q_\tau[z_2|z_1]\big] = Q_\tau\big[Q_\tau\big[(z_1 + z_2)|z_1\big]\big].$$

This should be the DM's objective function to achieve dynamic consistency in this example. Under independence, the above reduces to $Q_\tau[z_1] + Q_\tau[z_2]$.

In the general case, in period t, x_t is the state, x_{t+1} is the choice, and z_t is the shock, the period utility is $u(x_t, x_{t+1}, z_t)$, and the objective function should be

$$Q_\tau\left[\cdots Q_\tau\left[\sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}, z_t)\middle| z^{T-1}\right]\cdots \middle| z^1 = 1\right], \tag{8}$$

where there are T operators Q_τ above. This concludes the definition of the quantile preference.[16]

Of course, the above construction is heuristic and informal. A formal derivation of this preference requires an axiomatization, which is briefly discussed below. Our objective here is simply to offer a concrete setting where our preference is compelling.

2.3.2. *Axiomatization*

The preference defined by (8) is formally axiomatized in de Castro and Galvao (2018). In this section, we briefly discuss the main aspects of such axiomatization, which is divided into static and recursive (dynamic) parts.

In the static case, de Castro and Galvao (2018) followed very closely the developments of Chambers (2007).[17] The three main axioms that provide the quantile utility representation are monotonicity, ordinal covariance, and betting consistency (see, e.g., Chambers (2005, 2007)). Monotonicity stipulates that if, for each state of nature, the consequence of some act f is preferred to that of another act g, then f is preferred to g. This is a very standard property, valid for most preferences. Ordinal covariance requires that increasing transformations of a pair of acts do not change their ranking. In a sense, this axiom captures the property stated in (4). Finally, betting consistency allows writing the aggregation operator as a quantile with respect to some probability measure. These axioms are the building block to construct static quantile preferences.

Regarding the recursive case, de Castro and Galvao (2018) built on the results in Bommier, Kochov, and Le Grand (2017), who showed that a preference on a dynamic setting satisfying the axioms of weak order, continuity, recursivity, history independence, and stationarity can be represented by a utility function U that obeys the following recursive equation:

$$U(h) = W\big(h_0, I\big[U \circ h^1\big]\big), \tag{9}$$

where W is a time aggregator, I is a certainty equivalent functional, and h_0 and h^1 represent, respectively, the outcome of plan h in period 0 and h^1 its continuation from

[16]In the case that z_t's are independent, the above simplifies to $\sum_{t=0}^{T} \beta^t Q_\tau[u(x_t, x_{t+1}, z_t)]$.

[17]The settings in Chambers (2009) and Rostek (2010) are not used because the former works in a context of risk, instead of uncertainty, while the latter requires an infinite state space, which is not convenient for the axiomatization of the dynamic preferences.

period 1. This is contained in their Lemma 1. Bommier, Kochov, and Le Grand (2017, Proposition 1) then showed that if the preference additionally satisfies the axioms of monotonicity for deterministic prospects, and monotonicity, then W has an additively separable structure, that is, $W(c, x) = u(c) + \beta x$ or $W(c, x) = u(c) + b(c)x$.[18] The first case implies, in particular, that the decision maker evaluates riskless consumer streams according to the standard exponential discounting utility $\sum_{t=0}^{\infty} \beta^t u(c_t)$ required above in (6).[19] Given these results, de Castro and Galvao (2018) adapted the static quantile axioms to a dynamic setting and combined them with Bommier, Kochov, and Le Grand's (2017) axioms to show that the certainty equivalent $\mathrm{I}[\cdot]$ above is a quantile operator and that $W(c, x) = u(c) + \beta x$, thus obtaining the recursive equation

$$V(h) = u(h_0) + \beta \mathrm{Q}_\tau\big[V\big(h^1\big)\big]. \tag{10}$$

Section 3.3 below shows that this recursive structure leads to the preference definition (8).

It is useful to compare the justification of the preference in Section 2.3.1 with this axiomatization. In the example above, we first justified the quantile objective function. The parallel here is the axiomatization of the static quantile function obtained with Chambers's (2007) axioms (monotonicity, ordinal covariance, and betting consistency). The next step in Section 2.3.1 was to justify the standard exponential discounting for riskless streams (6), which is a consequence of Bommier, Kochov, and Le Grand's (2017) axioms of monotonicity for deterministic prospects and monotonicity. Finally, Section 2.3.1 discusses dynamic consistency, which is directly related to the recursive structure (9), implied by the other axioms of Bommier, Kochov, and Le Grand (2017). Section 3.3 below departs from this recursive structure to finally obtain the preferences defined by (8).

2.4. *Discussion*

In this subsection, we discuss other advantages and features of the quantile preferences that may be useful to further motivate its use.

2.4.1. *Other Advantages of the Static Quantile Preference*

Rostek (2010) discussed different motivations for the static quantile preference, among which are robustness, invariance with respect to ordinal transformations of the consequences, and properties of risk attitudes. She also discussed the use of quantile preferences for policy making and scenario-based analysis, and illustrated how quantile objective functions are useful for poll design and modeling risk attitude in an insurance setting (Medicare).

The separation of tastes and beliefs, which is a desirable property of preferences as discussed by Ghirardato, Maccheroni, and Marinacci (2005), can be added as a motivation. Indeed, economists often operate under an implicit assumption that the tastes of a decision maker are relatively stable, while beliefs change with the availability of new information. Ghirardato, Maccheroni, and Marinacci (2005) attempted to offer a result

[18]Bommier, Kochov, and Le Grand's (2017) axiom D6 has a strict monotonicity requirement built into it. Indeed, a time aggregator like $W(c, x) = \min\{c, x\}$ would not satisfy it. Notice that while the time aggregator is strictly monotonic, the quantile is only weakly monotonic.

[19]This time attitude with respect to deterministic prospects is usually directly imposed with an axiom. See, for instance, Epstein and Schneider (2003), Maccheroni, Marinacci, and Rustichini (2006), and Klibanoff, Marinacci, and Mukerji (2009).

with this separation, but were able to obtain only a partial separation with a certainty independence axiom. Quantile preferences allow for full separation between tastes and beliefs. Namely, the utility attributed to the tastes does not interfere with the beliefs part of the preference, which is robust to any monotonic transformation (and not only affine ones).

2.4.2. *Risk Attitude in the Quantile Model*

Manski (1988) and Rostek (2010) argued that the risk attitude of a quantile maximizer can be captured by τ. Manski (1988) discussed a definition of risk, which is based on the single-crossing property and allows one utility distribution to be termed riskier than another.

DEFINITION 2.1: We say that F_X is risker than F_Y if F_Y crosses F_X from below. That is,

$$F_Y(y) \le F_X(y), \quad \forall y < x \quad \text{and} \quad F_Y(y) \ge F_X(y), \quad \forall y > x. \tag{11}$$

Figure 2 illustrates (11), that is, that F_Y crosses F_X from below at x. Notice that X is more widespread than Y, which justifies the notion F_X is riskier than F_Y. Let us also illustrate this idea with a simple example. Let X be uniform on $[0, 1]$ and Y, uniform on $[\frac{1}{4}, \frac{3}{4}]$. These distributions have the same mean and the former has a larger variance, which justifies the notion that X is riskier than Y. It is easy to see that $\mathrm{Q}_\tau[X] = \tau$ and $\mathrm{Q}_\tau[Y] = \frac{\tau}{2} + \frac{1}{4}$. Therefore, $\mathrm{Q}_\tau[Y] > \mathrm{Q}_\tau[X]$ if $\tau < 0.5$, that is, a quantile maximizer with small τ prefers the less risky lottery Y. In contrast, if $\tau > 0.5$, the riskier X is preferred. In other words, a higher τ corresponds to more "propensity" for risk.

This property generalizes to quantiles in the following sense. Assume that F_Y crosses F_X from below at exactly one point (x), as shown in Figure 2. Thus, X can be intuitively considered "riskier" than Y. Notice that a τ-quantile maximizer (QM) prefers Y if $\tau < F_Y(x) = F_X(x)$, and a τ'-QM prefers X if $\tau' > F_Y(x) = F_X(x)$, as the figure also illustrates. Thus, a τ'-QM has more "propensity for risk" than a τ-QM for $\tau < \tau'$.[20] This property was used by Rostek (2010) to define a notion of "more risk aversion." See Rostek (2010, Section 6.1) and Manski (1988, Section 5) for further discussion. This notion can be extended to the dynamic setting in a natural way; see de Castro and Galvao (2018) for more details.

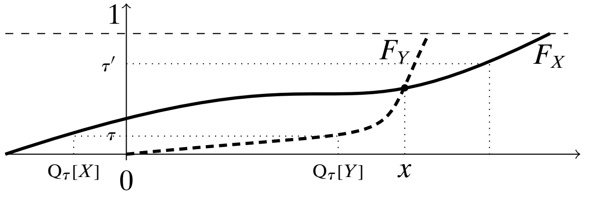


FIGURE 2.—F_Y crosses F_X from below once.

[20]We refrain from saying that the τ'-quantile maximizer is more risk averse to avoid confusion with risk aversion in the expected utility setting.

2.4.3. *Timing of the Resolution of Uncertainty and Intertemporal Substitution*

Since Kreps and Porteus (1978) and Epstein and Zin (1989), it is well understood that a separation of risk and intertemporal attitudes is possible only if the timing of the resolution of uncertainty matters. More recently, Bommier, Kochov, and Le Grand (2017, Proposition 3) showed that scale-invariant certainty equivalents generate what is called restricted indifference toward the timing of the resolution of uncertainty. This is, in a sense, the weakest form of indifference toward the timing of the resolution of uncertainty that still accommodates the separation of risk and intertemporal substitution attitudes. Since the quantile certainty equivalent operator is scale-invariant, it belongs to this selected class, and thus allows for this separation.

We illustrate how this separation can be achieved. Consider the utility index $u(c) = c^\rho$. If $\rho \in (0, 1)$, this corresponds to the case of risk aversion in the expected utility model, and if $\rho^1 < \rho^2$, 1 is more risk averse than 2, in the sense of having a higher coefficient of relative risk aversion. However, in the static quantile preferences, any $\rho > 0$ leads to exactly the same choices, as discussed above; see (4). In other words, the parameter ρ does not capture any aspect of the decision maker attitude toward risk. As discussed in Section 2.4.2, this attitude is captured by τ in both the static and dynamic cases.

If we have multiple periods, however, the parameter ρ plays an important role. Indeed, consider equation (10) with the same utility index above, that is,

$$V(c_0, \tilde{c}_1) = c_0^\rho + \beta \mathrm{Q}_\tau\big[\tilde{c}_1^\rho\big].$$

Applied to a deterministic prospect, that is, $\tilde{c}_1 = c_1$, this yields $c_0^\rho + \beta c_1^\rho$. It is easy to see that the elasticity of intertemporal substitution (EIS) in this case is simply $\frac{1}{1-\rho}$. Section 4.4 illustrates how to estimate the EIS with our dynamic quantile model using standard methods.

It is useful to compare our method with the most widely used method to separate risk aversion and the EIS, which is the following specification of Epstein and Zin (1989) and Weil (1990), with $\rho \neq 0$, $\alpha \neq 0$:

$$V^{EZ}(c_0, \tilde{c}_1) = \big(c_0^\rho + \beta\big(\mathrm{E}\big[\tilde{c}_1^\alpha\big]\big)^{\frac{\rho}{\alpha}}\big)^{\frac{1}{\rho}}.$$

As observed by Bommier, Kochov, and Le Grand (2017), this model satisfies monotonicity if and only if $\rho = \alpha$, in which case the model collapses to the standard expected utility model where the separation of risk aversion and EIS is not possible. In other words, for achieving its goal, the popular Esptein–Zin–Weil preferences are necessarily nonmonotonic. Bommier, Kochov, and Le Grand (2017, Lemmas 2 and 3) illustrated some of the problems that arise from this lack of monotonicity. In short, an Epstein–Zin–Weil decision maker may prefer to reduce lifetime utility in all states of the world just out of his willingness to reduce risk. In contrast, the willingness to reduce risk by a decision maker with monotonic preferences will never lead to reduce lifetime utility in all states of the world. This seems, therefore, a shortcoming of those preferences. Since dynamic quantile preferences are monotonic, they are immune to this criticism.

3. ECONOMIC MODEL AND THEORETICAL RESULTS

This section describes a dynamic economic model and develops a dynamic program theory for quantile preferences. The main results in this section are generalizations to the quantile preferences' case of the corresponding ones in Stokey, Lucas, and Prescott

(1989), which focus on expected utility. Our results increase the scope of potential applications of economic models substantially by using quantile utility. Moreover, the generalizations are of independent interest. The demonstrations are not routine since quantiles do not possess several of the convenient properties of expectations, such as linearity and the law of iterated expectations.

3.1. *Dynamic Model*

The recursive equation (10) describes the preference from one period to the next; it is now necessary to show how it determines the preferences over plans. To do this, we need some definitions and new notation.

3.1.1. *States and Shocks*

Let $\mathcal{X} \subset \mathbb{R}^p$ denote the state space, and $\mathcal{Z} \subseteq \mathbb{R}^k$ the range of the shocks (random variables) in the model. Let $x_t \in \mathcal{X}$ and $z_t \in \mathcal{Z}$ denote, respectively, the state and the shock in period t, both of which are known by the decision maker at the beginning of period t. We may omit the time indexes for simplicity, when it is convenient. Let $\mathcal{Z}^t = \mathcal{Z} \times \cdots \times \mathcal{Z}$ (t times, for $t \in \mathbb{N}$), $\mathcal{Z}^\infty = \mathcal{Z} \times \mathcal{Z} \times \cdots$, and $\mathbb{N}^0 \equiv \mathbb{N} \cup \{0\}$. Given $z \in \mathcal{Z}^\infty$, $z = (z_1, z_2, \ldots)$, we denote $(z_t, z_{t+1}, \ldots)$ by ${}_tz$ and $(z_t, z_{t+1}, \ldots, z_{t'})$ by ${}_tz_{t'}$. A similar notation can be used for $x \in \mathcal{X}^\infty$.

The random shocks will follow a time-invariant (stationary) Markov process. More precisely, a probability density function (p.d.f.) $f : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$ establishes the dependence between Z_t and Z_{t+1}, such that the process is invariant with respect to t. For simplicity of notation, we will frequently represent Z_t and Z_{t+1} by Z and Z', respectively.[21] We will assume that f and $\mathcal{Z}$ satisfy standard assumptions, as explicitly stated below in Section 3.2.

For any topological space $\mathcal{S}$, we will denote by $\sigma(\mathcal{S})$ the Borel σ-algebra. For each $z \in \mathcal{Z}$ and $A \in \sigma(\mathcal{Z})$, define

$$K(z, A) \equiv \int_A f\left(z'|z\right) dz',$$

where $f(z'|z) = \frac{f(z,z')}{\int_{\mathcal{Z}} f(z,z')\,dz'}$. Thus, K is a probabilistic kernel, that is, (i) $z \mapsto K(z, A)$ is measurable for every $A \in \sigma(\mathcal{Z})$; and (ii) $A \mapsto K(z, A)$ is probability measure for every $z \in \mathcal{Z}$. In other words, K represents a conditional probability, and we may emphasize this fact by writing $K(A|z)$ instead of $K(z, A)$. We will also abuse notation by denoting $K(z, \{\tilde{z} : \tilde{z} \leq z'\})$ simply by $K(z'|z)$.

3.1.2. *Plans*

At the beginning of period t, the decision maker knows the current state x_t and learns the shock z_t and decides (according to preferences defined below) the future state $x_{t+1} \in \Gamma(x_t, z_t) \subset \mathcal{X}$, where $\Gamma(x, z)$ is the constraint set.[22] From this, we can define plans as follows:

[21]This structure implies, in particular, that the information filtration is fixed throughout.

[22]This model is very close to the one discussed in Stokey, Lucas, and Prescott (1989, Chapter 9). There are different, slightly more complicated dynamic models where the state is not chosen by the decision maker, but defined by the shock. Stokey, Lucas, and Prescott (1989, Chapter 9) discussed how their results can be adapted to those models. In our setup, this extension may be more involved.

DEFINITION 3.1: A plan h is a profile $h = (h_t)_{t\in\mathbb{N}}$ where, for each $t \in \mathbb{N}$, h_t is a measurable function from $\mathcal{X} \times \mathcal{Z}^t$ to $\mathcal{X}$.[23] The set of plans is denoted by H.

The interpretation of the above definition is that a plan $h_t(x_t, z^t)$ represents the choice that the individual makes at time t upon observing the current state x_t and the sequence of previous shocks z^t. The following notation will simplify statements below.

DEFINITION 3.2: Given a plan $h = (h_t)_{t\in\mathbb{N}} \in H$, $x \in \mathcal{X}$, and realization $z^\infty = (z_1, \dots) \in Z^\infty$, the sequence associated to (x, z^∞) is the sequence $(x_t^h)_{t\in\mathbb{N}^0} \in \mathcal{X}^\infty$ defined recursively by $x_1^h = x$ and $x_t^h = h_{t-1}(x_{t-1}^h, z^{t-1})$, for $t \geq 2$. Similarly, given $h \in H$, $(x, z^t) \in \mathcal{X} \times Z^t$, the t-sequence associated to (x, z^t) is $(x_l^h)_{l=1}^t \in \mathcal{X}^t$ defined recursively as above.

We may write $x_t^h(\cdot)$, $x_t^h(x, z^t)$, or $x_t^h(x, z^\infty)$ to emphasize that x_t^h depends on the initial state x and on the sequence of shocks z^∞, up to time t.

DEFINITION 3.3: A plan h is feasible from $(x, z) \in \mathcal{X} \times \mathcal{Z}$ if $h_t(x_t^h, z^t) \in \Gamma(x_t^h, z_t)$ for every $t \in \mathbb{N}$ and $z^\infty \in \mathcal{Z}^\infty$ such that $x_1^h = x$ and $z_1 = z$.

We denote by $H(x, z)$ the set of feasible plans from $(x, z) \in \mathcal{X} \times \mathcal{Z}$. Let H denote the set of all feasible plans from some point, that is, $H \equiv \bigcup_{(x,z)\in\mathcal{X}\times\mathcal{Z}} H(x, z)$.

3.1.3. *Preferences*

Let Ω_t represent all the information revealed up to time t.[24] We assume that in time t with revealed information Ω_t, the consumer/decision maker has a preference $\succcurlyeq_{t,\Omega_t}$ over plans $h, h' \in H(x, z)$, which is represented by a function $V_t : H \times \mathcal{X} \times Z^t \to \mathbb{R}$, that is,

$$h' \succcurlyeq_{t,x,\Omega_t} h \quad \iff \quad V_t(h', x, z^t) \geq V_t(h, x, z^t). \tag{12}$$

Notice that the preferences in (12) are time, information, and state contingent.

A special case of this model corresponds to the standard case of expected utility, that is,

$$V_t(h, x, z^t) = \mathrm{E}\left[\sum_{s\geq t} \beta^{s-t} u(x_s^h, x_{s+1}^h, Z_s) \middle| Z^t = z^t\right], \tag{13}$$

where $u : \mathcal{X} \times \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ is the current-period utility function. That is, $u(x, y, z)$ denotes the instantaneous utility obtained in the current period when $x \in \mathcal{X}$ denotes the current state, $y \in \mathcal{X}$, the choice in the current state, and $z \in \mathcal{Z}$, the current shock.

A first attempt to define the dynamic quantile preference would be to substitute the expectation operator E by the quantile operator Q_τ in (13), that is,

$$V_t(h, x, z^t) = \mathrm{Q}_\tau\left[\sum_{s\geq t} \beta^{s-t} u(x_s^h, x_{s+1}^h, Z_s) \middle| Z^t = z^t\right]. \tag{14}$$

The preference defined by (14) may be referred to as simple quantile preference. Although this seems the natural way to adapt the standard definition, this would lead to

[23] In the expressions below, $h_0(z^0)$ should be understood as just $h_0 \in \mathcal{X}$.

[24] With the knowledge of a fixed h, Ω_t reduces to the initial state x_1 and the sequence of shocks z^t. More generally, we could take the sequence of states and shocks (x^t, z^t).

dynamically inconsistent preferences, because the analog of the "law of iterated expectations" does not hold for quantiles; see Example 3.7 below. Instead, we will need to take a different route. Note that the functions V_t defined by (13) satisfy the following recursive equation:

$$V_t(h, x, z^t) = u(x_t^h, x_{t+1}^h, z_t) + \beta \mathrm{E}[V_{t+1}(h, x, (z^t, Z_{t+1})) | Z^t = z^t]. \tag{15}$$

We adapt equation (15) by replacing the expectation operator E with the quantile operator Q_τ, that is, we impose

$$V_t(h, x, z^t) = u(x_t^h, x_{t+1}^h, z_t) + \beta \mathrm{Q}_\tau[V_{t+1}(h, x, (Z^t, z_{t+1})) | Z^t = z^t]. \tag{16}$$

The recursive equation (16) is the building block of our dynamic quantile preferences and it leads to dynamically consistent preferences, as we show below. This is the reason why Section 2 provided a detailed justification of recursive equation (16) in the simpler form of (10).

In Section 3.3 below, we explicitly define a sequence of functions V_t that satisfy (16) and will specify the preferences (12). Nevertheless, before we provide an additional formalization for the definition of the sequence of recursive functions, it is useful to build intuition on how the recursive equation (16) leads to an expression in quantiles that would be different from the expected utility case, developed from (15).

To see this, let us adopt $t = 1$ and substitute the expression of $V_{t+1} = V_2$ by the expression in (15) and use superscript E to denote the expected utility case; we obtain

$$V_1^{\mathrm{E}}(h, x, z^t) = u(x_1^h, x_2^h, z_1) + \beta \mathrm{E}[u(x_2^h, x_3^h, z_2) + \beta \mathrm{E}[V_2^{\mathrm{E}}(h, x, z^t) | Z_2 = z_2] | Z_1 = z].$$

Above, we could eliminate the expectation with respect to $Z_2 = z_2$ using the law of iterated expectations. Since the same simplification is not possible in the quantile case, we will avoid it here. Moreover, we are able to put all the terms inside the expectations. That is, we can write

$$\begin{aligned}
&V_1^{\mathrm{E}}(h, x, z^t) \\
&\quad = \mathrm{E}[\mathrm{E}[u(x_1^h, x_2^h, z_1) + \beta u(x_2^h, x_3^h, z_2) + \beta^2 V_2^{\mathrm{E}}(h, x, z^t) | Z_2 = z_2] | Z_1 = z] \\
&\quad = \mathrm{E}\left[\mathrm{E}\left[\mathrm{E}\left[\sum_{t=1}^{3} \beta^{t-1} u(x_t^h, x_{t+1}^h, z_t) + \beta^3 V_3^{\mathrm{E}}(h, x, z^t) \middle| Z_3 = z_3\right] \middle| Z_2 = z_2\right] \middle| Z_1 = z\right] \\
&\quad = \mathrm{E}\left[\cdots \mathrm{E}\left[\sum_{t=1}^{n} \beta^{t-1} u(x_t^h, x_{t+1}^h, z_t) + \beta^n V_n^{\mathrm{E}}(h, x, z^t) \middle| Z_n = z_n\right] \middle| \cdots \middle| Z_1 = z\right],
\end{aligned}$$

where there are n expectation operators E and corresponding conditionals $Z_t = z_t$ in the last line above. Following the same developments from (16), we obtain

$$\begin{aligned}
&V_1^{\mathrm{Q}_\tau}(h, x, z^t) \\
&\quad = u(x_1^h, x_2^h, z_1) + \beta \mathrm{Q}_\tau[u(x_2^h, x_3^h, z_2) + \beta \mathrm{Q}_\tau[V_2^{\mathrm{Q}_\tau}(h, x, z^t) | Z_2 = z_2] | Z_1 = z] \\
&\quad = \mathrm{Q}_\tau[\mathrm{Q}_\tau[u(x_1^h, x_2^h, z_1) + \beta u(x_2^h, x_3^h, z_2) + \beta^2 V_2^{\mathrm{Q}_\tau}(h, x, z^t) | Z_2 = z_2] | Z_1 = z]
\end{aligned}$$

$$= \mathrm{Q}_\tau\left[\mathrm{Q}_\tau\left[\mathrm{Q}_\tau\left[\sum_{t=1}^{3}\beta^{t-1}u\left(x_t^h, x_{t+1}^h, z_t\right) + \beta^3 V_3^{\mathrm{Q}_\tau}\left(h, x, z^t\right)\Big| Z_3 = z_3\right]\Big| Z_2 = z_2\right]\Big| Z_1 = z\right]$$

$$= \mathrm{Q}_\tau\left[\cdots \mathrm{Q}_\tau\left[\sum_{t=1}^{n}\beta^{t-1}u\left(x_t^h, x_{t+1}^h, z_t\right) + \beta^n V_n^{\mathrm{Q}_\tau}\left(h, x, z^t\right)\Big| Z_n = z_n\right]\Big| \cdots \Big| Z_1 = z\right], \quad (17)$$

where the operator $\mathrm{Q}_\tau[\cdot]$ and corresponding conditionals $Z_t = z_t$ appear n times in the last line above. In order to simplify the above equation, we will introduce the following notation:

$$\mathrm{Q}_\tau^n[\cdot] \equiv \mathrm{Q}_\tau\left[\cdots\left[\mathrm{Q}_\tau[\,\cdot\,|Z_n = z_n]\right|\cdots\right]|Z_1 = z\right], \quad (18)$$

where the operator Q_τ and corresponding conditionals appear n times. Therefore, by using the notation defined by (18), we are able to rewrite (17) as

$$V_1^{\mathrm{Q}_\tau}\left(h, x, z^t\right) = \mathrm{Q}_\tau^n\left[\sum_{t=1}^{n}\beta^{t-1}u\left(x_t^h, x_{t+1}^h, z_t\right) + \beta^n V_n^{\mathrm{Q}_\tau}\left(h, x, z^t\right)\right]. \quad (19)$$

The next step is to take the limit as n goes to ∞. The formalization of such limit will be made in Section 3.3 below, but one can now intuitively understand the following:

$$V_1^{\mathrm{Q}_\tau}\left(h, x, z^t\right) = \mathrm{Q}_\tau^\infty\left[\sum_{t=1}^{\infty}\beta^{t-1}u\left(x_t^h, x_{t+1}^h, z_t\right)\right], \quad (20)$$

as a notation for an (infinite) sequence of applications of $\mathrm{Q}_\tau^n[\,\cdot\,|Z^t = z^t]$.

Note that if we introduce an analogous notation of (18), that is, E^∞ for a(n infinite) sequence of conditional expectations, because of the law of iterated expectations, we obtain

$$V_1^{\mathrm{E}}\left(h, x, z^t\right) = \mathrm{E}^\infty\left[\sum_{t=1}^{\infty}\beta^{t-1}u\left(x_t^h, x_{t+1}^h, z_t\right)\right] = \mathrm{E}\left[\sum_{t=1}^{\infty}\beta^{t-1}u\left(x_t^h, x_{t+1}^h, z_t\right)\Big| Z_1 = z_1\right],$$

which is the expression in (13). Therefore, expression (20) is the corresponding generalization of (13): the difference, that is, the fact that we can substitute E^∞ by E but not Q_τ^∞ by Q_τ, is explained by whether or not the law of iterated expectations holds. Indeed, as Example 3.7 below shows, this law does not hold for quantiles.

It is worth mentioning that in the particular case in which the z_t are independent, (19) and (20) can be simplified. Notice that independence implies

$$\mathrm{Q}_\tau\left[u(x_t, x_{t+1}, z_t)|z_{t-1}\right] = \mathrm{Q}_\tau\left[u(x_t, x_{t+1}, z_t)\right],$$

which is a number, not a random variable. Being a number, it can be taken out of the quantile. Thus, (19) simplifies to

$$V_1^{\mathrm{Q}_\tau}\left(h, x, z^t\right) = \sum_{t=1}^{n}\beta^{t-1}\mathrm{Q}_\tau\left[u\left(x_t^h, x_{t+1}^h, z_t\right)\right] + \beta^n \mathrm{Q}_\tau\left[V_n^{\mathrm{Q}_\tau}\left(h, x, z^t\right)\right],$$

and (20) simplifies to

$$V_1^{\mathrm{Q}_\tau}\left(h, x, z^t\right) = \sum_{t=1}^{\infty}\beta^{t-1}\mathrm{Q}_\tau\left[u\left(x_t^h, x_{t+1}^h, z_t\right)\right].$$

3.2. *Assumptions*

Now we state the assumptions used for establishing the main results. We organize the assumptions in two groups. The first group collects basic assumptions, which will be assumed throughout the paper, even if they are not explicitly stated. The second group of assumptions will be used only to obtain special, desirable properties of the value function.

ASSUMPTION 1—Basic: *The following properties are maintained throughout the paper*:
(i) $\mathcal{Z} \subseteq \mathbb{R}^k$ *is convex*;
(ii) $f : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$ *is continuous*, *symmetric*, *and* $f(z, z') > 0$, *for all* $(z, z') \in \mathcal{Z} \times \mathcal{Z}$;[25]
(iii) $\mathcal{X} \subset \mathbb{R}^p$ *is convex*;
(iv) $u : \mathcal{X} \times \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *is continuous and bounded*;
(v) *The correspondence* $\Gamma : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{X}$ *is continuous*, *with nonempty*, *compact*, *and convex values*.

Note that in the above assumption, property (i) allows an unbounded multidimensional Markov process, requiring only that the support is convex. Property (ii) imposes continuity of f, the p.d.f. that establishes the dependence between Z_t and Z_{t+1}, and requires it to be strictly positive in the support of the Markov process, $\mathcal{Z}$. The state space $\mathcal{X}$ is not required to be compact, but only convex by property (iii). Property (iv) is the standard continuity assumption. Property (v) and the continuity of u required in property (iv) guarantee that an optimal choice always exist.

For some results, we will also require differentiability, concavity, and monotonicity assumptions.

ASSUMPTION 2—Differentiability, Concavity, and Monotonicity: *The following hold*:
(i) $\mathcal{Z} \subseteq \mathbb{R}$ *is an interval*;
(ii) *If* $h : \mathcal{Z} \to \mathbb{R}$ *is weakly increasing and* $z \leq z'$, *then*

$$\int_{\mathcal{Z}} h(\alpha) f(\alpha|z)\, d\alpha \leq \int_{\mathcal{Z}} h(\alpha) f(\alpha|z')\, d\alpha; \tag{21}$$

(iii) $u : \mathcal{X} \times \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *is* C^1, *strictly concave in the first two variables and strictly increasing in the last variable*;
(iv) *For every* $x \in \mathcal{X}$ *and* $z \leq z'$, $\Gamma(x, z) \subseteq \Gamma(x, z')$;
(v) *For all* $z \in \mathcal{Z}$ *and all* $x, x' \in \mathcal{X}$, $y \in \Gamma(x, z)$ *and* $y' \in \Gamma(x', z)$ *imply*

$$\theta y + (1 - \theta) y' \in \Gamma[\theta x + (1 - \theta) x', z], \quad \text{for all } \theta \in [0, 1].$$

To work with monotonicity, we restrict the dimension of the Markov process to $k = 1$ in Assumption 2(i). Assumptions 2(ii)–2(v) are standard conditions on dynamic models (see, e.g., Assumptions 9.8–9.15 in Stokey, Lucas, and Prescott (1989)). Assumption 2(ii) implies that whenever $z \leq z'$,

$$K(w|z') = \int_{\{\alpha \in \mathcal{Z} : \alpha \leq w\}} f(\alpha|z')\, d\alpha \leq \int_{\{\alpha \in \mathcal{Z} : \alpha \leq w\}} f(\alpha|z)\, d\alpha = K(w|z), \tag{22}$$

[25] Symmetry guarantees stationarity since $\Pr([Z_1 \in A]) = \int_{\mathcal{Z}} \int_A f(z_1, z_2)\, dz_1\, dz_2 = \int_A \int_{\mathcal{Z}} f(z_1, z_2)\, dz_1\, dz_2 = \Pr([Z_2 \in A])$.

for all w.[26] In other words, $K(\cdot|z')$ first-order stochastically dominates $K(\cdot|z)$. Assumption 2(iii) allows us to establish the continuity and differentiability of the value function. Assumption 2(iv) only requires the monotonicity of the choice set. Assumption 2(v) implies that $\Gamma(x,z)$ is a convex set for each $(x,z)\in\mathcal{X}\times\mathcal{Z}$, and that there are no increasing returns.

It should be noted that monotonicity is also important for econometric reasons. Indeed, Matzkin (2003, Lemma 1, p. 1345) showed that two econometric models are observationally equivalent if and only if there are strictly increasing functions mapping one to another. Thus, in a sense, the quantile implied by a model is the essence of what can be identified by an econometrician.

3.3. *The Sequence of Recursive Functions*

In this subsection, we define the sequence of functions V_t that satisfy (16) and specify the preferences (12). For this, we need to define a transformation. Let $\mathcal{L}$ be the set of real-valued functions from $\mathcal{X}\times\mathcal{Z}$ to $\mathbb{R}$ and let $\mathcal{C}\subset\mathcal{L}$ denote the set of bounded continuous functions from $\mathcal{X}\times\mathcal{Z}$ to $\mathbb{R}$, endowed with the sup norm. It is well known that $\mathcal{C}$ is a Banach space. Let us fix $h\in H$ and $\tau\in(0,1)$, and define the transformation $\mathbb{T}^h:\mathcal{C}\to\mathcal{L}$ (the dependency on τ is omitted) by the following:

$$\mathbb{T}^h(V)(x,z)=u\big(x_1^h,x_2^h,z_1\big)+\beta\mathrm{Q}_\tau\big[V\big(x_2^h,Z_2\big)|Z_1=z\big],$$

where $(x_1^h,z_1)=(x,z)$ and $x_2^h=h(x,z)$. We show that the image of $\mathbb{T}^h$ is indeed in $\mathcal{C}$ continuous and that $\mathbb{T}^h$ is a contraction and, therefore, has a unique fixed point:

THEOREM 3.4: *$\mathbb{T}^h(V)$ is continuous and bounded on $\mathcal{X}\times\mathcal{Z}$, that is, $\mathbb{T}(\mathcal{C})\subset\mathcal{C}$. Moreover, $\mathbb{T}^h$ is a contraction and has a unique fixed point, denoted $V^h\in\mathcal{C}$.*

Now we can define V_t as follows:

$$V_t\big(h,x,z^t\big)=V^h\big(x_t^h,z_t\big),$$

where $(x_l^h)_{l=1}^t$ is the associated t-sequence to (x,z^t) (see Definition 3.2). From the fact that V^h is the unique fixed point of $\mathbb{T}^h$, it is clear that (16) holds. This completes the definition of the preferences (12).

It is possible to write V^h in a more explicit form. For this, let us define

$$\begin{aligned}V^{h,n}(x,z)&=u\big(x_1^h,x_2^h,z_1\big)+\mathrm{Q}_\tau\big[\beta u\big(x_2^h,x_3^h,z_2\big)+\mathrm{Q}_\tau\big[\beta^2u\big(x_3^h,x_4^h,z_3\big)+\cdots\\&\qquad+\mathrm{Q}_\tau\big[\beta^nu\big(x_{n+1}^h,x_{n+2}^h,z_n\big)|Z_n=z_n\big]\cdots|Z_1=z\big]\\&=\mathrm{Q}_\tau\left[\cdots\left[\mathrm{Q}_\tau\left[\sum_{t=0}^n\beta^tu\big(x_{t+1}^h,x_{t+2}^h,z_t\big)\middle|Z_n=z_n\right]\cdots\right]\middle|Z_1=z\right]\\&=\mathrm{Q}_\tau^n\left[\sum_{t=0}^n\beta^tu\big(x_{t+1}^h,x_{t+2}^h,z_t\big)\right],\end{aligned}$$

[26]To obtain (22), it is enough to use $h(z)=-1_{\{\alpha\in\mathcal{Z}:\alpha\le w\}}(z)$ in (21).

where the expression $Q^n_\tau[\cdot]$ in the last line is just a short notation for the conditional quantiles applied successively, as shown in the previous line; see (18). With this definition, we obtain the following:

PROPOSITION 3.5: $V^h(x,z) = \lim_{n\to\infty} V^{h,n}(x,z)$.

Thus, the existence of the limit $\lim_{n\to\infty} V^{h,n}(x,z)$ allows us to define the notation $Q^\infty_\tau[\cdot]$, that is,

$$
\begin{aligned}
V^h(x,z) &= Q^\infty_\tau\left[\sum_{t=0}^{\infty} \beta^t u\left(x^h_t, x^h_{t+1}, z_t\right)\right] \\
&= u\left(x^h_1, x^h_2, z_1\right) + Q_\tau\left[\beta u\left(x^h_2, x^h_3, z_2\right) + Q_\tau\left[\beta^2 u\left(x^h_3, x^h_4, z_3\right) + \cdots\right.\right. \\
&\quad \left.\left. + Q_\tau\left[\beta^n u\left(x^h_{n+1}, x^h_{n+2}, z_{n+1}\right) + \cdots | \cdots\right] \cdots | Z_2 = z_2\right] | Z_1 = z\right]. \qquad (23)
\end{aligned}
$$

We turn now to verify that this preference is dynamically consistent.

3.4. *Dynamic Consistency*

Our objective is to develop a dynamic theory for quantile preferences. Thus, the dynamic consistency of such preferences is of uttermost importance. In this subsection, we formally define dynamic consistency and show that it is satisfied by the above defined preferences. The following definition comes from Maccheroni, Marinacci, and Rustichini (2006); see also Epstein and Schneider (2003).

DEFINITION 3.6—Dynamic Consistency: The system of preferences $\succcurlyeq_{t,\Omega_t}$ is *dynamically consistent* if, for every t and Ω_t and for all plans h and h', $h_{t'}(\cdot) = h'_{t'}(\cdot)$ for all $t' \leq t$ and $h' \succcurlyeq_{t+1,\Omega'_{t+1},x} h$ for all Ω'_{t+1}, x, implies $h' \succcurlyeq_{t,\Omega_t,x} h$.

One shows dynamic consistency of the expected utility preferences by appealing to the law of iterated expectations. Unfortunately, an analogue of such law does not hold for quantiles, as the following example shows.

EXAMPLE 3.7—Law of Iterated Quantiles: We will define a random variable X such that

$$
Q_\tau\left[Q_\tau[X|\Sigma_1]|\Sigma_0\right] \neq Q_\tau[X|\Sigma_0], \qquad (24)
$$

where $\Sigma_1 \supset \Sigma_0$ are two σ-algebras on Ω.

Let $\Omega = \{1,2,3,4\}$ and $P(\{\omega\}) = 1/4$ for all $\omega \in \Omega$. Define $\Sigma_0 = \{\emptyset, \Omega\}$ and $\Sigma_1 = \{\emptyset, E_1, E_2, \Omega\}$, where $E_1 = \{1,2\}$ and $E_2 = \{3,4\}$. Let $X(\omega) = \omega$. Then, for $\tau \in (0.5, 0.75)$,

$$
Q_\tau[X|\Sigma_1]_\omega = \begin{cases} 2 & \text{if } \omega \in E_1, \\ 4 & \text{if } \omega \in E_2. \end{cases}
$$

Therefore, $Q_\tau[Q_\tau[X|\Sigma_1]|\Sigma_0] = 4$ but $Q_\tau[X|\Sigma_0] = Q_\tau[X] = 3$, which establishes (24).

As one could expect, such failure suggests that quantile preferences would be dynamically inconsistent. This is in fact the case if we adopt the simple quantile preferences defined by (14). The following example illustrates that dynamic consistency fails for such preferences. It is very related to the example of Section 2.3.1.

EXAMPLE 3.8—Reversal of Choices for Simple Quantiles: There exist random variables X and Y and σ-algebras $\Sigma_1 \supset \Sigma_0$ on Ω, such that

$$\begin{aligned} &\mathrm{Q}_\tau[X|\Sigma_1]_{(\omega)} \geq \mathrm{Q}_\tau[Y|\Sigma_1]_{(\omega)}, \quad \forall \omega \in \Omega, \quad \text{but} \\ &\mathrm{Q}_\tau[X|\Sigma_0]_{(\omega)} < \mathrm{Q}_\tau[Y|\Sigma_0]_{(\omega)}, \quad \forall \omega \in \Omega. \end{aligned} \tag{25}$$

It is enough to define X and Y as, respectively, U^{blue} and U^{red} of the example at the end of Section 2.3.1. Let $\Omega = [0, 4]$, $\Sigma_0 = \{\emptyset, \Omega\}$, and Σ_1 be generated by the partition $\{E_1, E_2\}$, where $E_1 = [0, 2)$ and $E_2 = [2, 4]$. Thus, $E_1 = [0, 2)$ corresponds to the event $z_1 = 0$ and $E_2 = [2, 4)$ corresponds to the event $z_1 = 2$. From the discussion in Section 2.3.1, (25) holds.

Note that (25) suggests a potential negation of dynamic consistency for quantile preferences in general. Indeed, this failure would be fatal for dynamic consistency if we had chosen the preference to seek the maximization of $\mathrm{Q}_\tau[\sum_{t=0}^{\infty} \beta^t u(x_t^h, x_{t+1}^h, z_t)|Z_t = z_t]$, because changing from one period to the other could imply a reversion of choices, which is exactly what (25) illustrates. More formally, we can restate the first part of (25) for the simple quantile preference defined by (14) as follows:

$$\tilde{V}_0(X) = \mathrm{Q}_\tau[X|\Sigma_0] < \mathrm{Q}_\tau[Y|\Sigma_0] = \tilde{V}_0(Y),$$

while the second part establishes that $\forall \omega \in \Omega$,

$$\tilde{V}_1(X, \omega) = \mathrm{Q}_\tau[X|\Sigma_1]_{(\omega)} \geq \mathrm{Q}_\tau[Y|\Sigma_1]_{(\omega)} = \tilde{V}_1(Y, \omega).$$

This proves that this preference is not dynamically consistent: the initial preference for Y is reversed at time 1 in all states of nature.

However, we have adopted as preference $\mathrm{Q}_\tau^\infty[\sum_{t=0}^{\infty} \beta^t u(x_t^h, x_{t+1}^h, z_t)]$, which involves an infinite sequence of nested conditional quantiles. This is discussed in Section 3.1.3, where the notation $\mathrm{Q}_\tau^\infty[\cdot]$ is introduced. This sequence is exactly what allows to obtain dynamic consistency. Indeed, in our framework, quantile preferences are dynamically consistent and amenable to the use of the standard techniques of dynamic programming, as the following result establishes.

THEOREM 3.9: *The quantile preferences defined by* (23) *are dynamically consistent.*

This result is important, because it implies that no money-pump can be used against a decision maker with quantile preferences. Many preferences that depart from the expected utility framework do not satisfy dynamic consistency.

To avoid this dynamic inconsistency, we adopted the iterated quantile preference (23). The following example illustrates Theorem 3.9 and how this recursive structure guarantees dynamic consistency.

EXAMPLE 3.10—Consistency of Choice for Iterated Quantiles: Now we will show that if instead of the simple quantile preference (14), we use (23), there is no choice reversal. Let τ, X, Y, and Σ_1, Σ_0 be as in Example 3.8 above and consider the preference defined by (23). We have

$$V_0(X) = \mathrm{Q}_\tau\big[\mathrm{Q}_\tau[X|\Sigma_1]|\Sigma_0\big] \geq \mathrm{Q}_\tau\big[\mathrm{Q}_\tau[Y|\Sigma_1]|\Sigma_0\big] = V_0(Y)$$

and, $\forall \omega \in \Omega$,

$$V_1(X, \omega) = \mathrm{Q}_\tau[X|\Sigma_1]_{(\omega)} \geq \mathrm{Q}_\tau[Y|\Sigma_1]_{(\omega)} = V_1(Y, \omega).$$

In other words, X is always preferred to Y and there is no reversal because the sequence of events is fixed and taken into account.[27]

Our approach to establish dynamic consistency is similar to that taken by Epstein and Schneider (2003) for the maximin expected utility dynamic preferences, in the sense that the filtration of events where decisions are made is fixed. As discussed by Strzalecki (2013, p. 1048), this is one of the main approaches that have been used to obtain dynamic consistency for different preferences.

We also note that Epstein and Le Breton (1993) essentially proved that dynamic consistent preferences are "probabilistic sophisticated" in the sense of Machina and Schmeidler (1992).[28] Probabilistic sophistication *roughly* means that the preference is "based" in a probability. Extending Machina–Schmeidler's definition, Rostek (2010) showed that the static quantiles preferences are probabilistic sophisticated for $\tau \in (0, 1)$. Her observation is also valid for our dynamic quantile preference. However, we do not use these developments, since Theorem 3.9 offers a direct proof of dynamic consistency.

3.5. *The Value Function*

In this subsection, we establish the existence of the value function associated to the dynamic programming problem for the quantile utility and some of its properties. This is accomplished through a contraction fixed point theorem.

The first step is to define the contraction operator; this is similar to what we have defined in Section 3.3. For $\tau \in (0, 1)$, define the transformation $\mathbb{M}^\tau : \mathcal{C} \to \mathcal{C}$ as

$$\mathbb{M}^\tau(v)(x, z) = \sup_{y \in \Gamma(x,z)} u(x, y, z) + \beta \mathrm{Q}_\tau\big[v(y, w)|z\big]. \tag{26}$$

Note that this is similar to the usual dynamic programming problem, in which the expectation operator $\mathrm{E}[\cdot]$ is in place of $\mathrm{Q}_\tau[\cdot]$. The main objective is to show that the above transformation has a fixed point, which is the value function of the dynamic programming problem. The following result establishes the existence of the contraction $\mathbb{M}^\tau$ under the basic assumptions assumed throughout this paper.

THEOREM 3.11: $\mathbb{M}^\tau$ *is a contraction and has a unique fixed point* $v^\tau \in \mathcal{C}$.

The unique fixed point of the problem will be the value function of the problem. Notice that the proof of this theorem is not just a routine application of the similar theorems from the expected utility case. In particular, the continuity of the function $(x, z) \mapsto \mathrm{Q}_\tau[v(x, w)|z]$ is not immediate as in the standard case. Since v is not assumed to be strictly increasing in the second argument, it can be constant at some level. Constant values may potentially lead to discontinuities in the c.d.f or quantile functions; see illustration in Section A.1 in the Appendix. We develop careful arguments and explore a strictly increasing property of $\mathrm{Q}_\tau[v(x, w)|z]$ together with continuity of v to establish the result; see the proof of Lemma A.5 for further details.

The next step is to establish the differentiability and monotonicity of the value function.

---

[27]In this paper, for simplicity, we assume a fixed filtration. This assumption might be unsuitable for information acquisition problems; see, for example, Berk (1997) and Skiadas (1998).

[28]See also Karni and Schmeidler (1991).

THEOREM 3.12: *If Assumption 2 holds, then* $v^\tau : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *is differentiable in* x, *strictly increasing in* z, *and strictly concave in an interior point* x. *Moreover,*

$$\frac{\partial v^\tau}{\partial x_i}(x, z) = \frac{\partial u}{\partial x_i}\big(x, y^*, z\big),$$

where $y^* \in \Gamma(x, z)$ *is the unique maximizer of* (26), *assumed to be interior to* $\Gamma(x, z)$.

This theorem delivers interesting and important properties of the value function. It establishes that the value function that one obtains from quantile functions possesses, essentially, the same basic properties of the value function of the corresponding expected utility problem. The second part of Theorem 3.12 is very important for the characterization of the problem. It is the extension of the standard envelope theorem for the quantile utility case. We adapt Benveniste and Scheinkman's (1979) argument for showing the concavity and differentiability of the value function in the expected utility case for the quantiles. Since the quantile function does not have the convenient properties of the expectation, Theorem 3.12's proof relies on the monotonicity of the quantiles together with the assumption that z is unidimensional (see Assumption 2). This unidimensionality requirement does not seem overly restrictive in most practical applications. For example, it allows us to tackle the standard intertemporal consumption model, as Section 4 shows.

REMARK 3.13: The result in Theorem 3.11 is related to that in Marinacci and Montrucchio (2010). They established the existence and uniqueness of the value function in a more general setup. Nevertheless, we are able to provide sharper characterizations of the fixed point v^τ. In particular, Theorem 3.11 establishes that v^τ is continuous, and Theorem 3.12, that it is differentiable, concave, and increasing.

3.6. *The Principle of Optimality*

This subsection establishes that the principle of optimality holds in our model (Proposition 3.17 below). That is, we show that optimizing period after period, as in the recursive problem (26), yields the same result as choosing the best plan for the whole horizon of the problem. In order to do that, we have to complete three tasks. First, we define the problem of choosing plans. Next, we revisit the recursive problem to establish a result that will be useful in the sequel. Finally, we show that choosing plans for the entire horizon and solving the problem step by step as in the recursive problem, lead to the same values.

Let us begin by establishing that the set of feasible plans departing from $(x, z) \in \mathcal{X} \times \mathcal{Z}$ at time t is nonempty. More formally, let us define

$$H_t(x, z) \equiv \big\{h \in H(x, z) : \exists\big(x, z^t\big) \in \mathcal{X} \times \mathcal{Z}^t, \text{with } z_t = z, \text{ such that } x_t^h\big(x, z^t\big) = x\big\}.$$

Thus, $H_1(x, z)$ is just $H(x, z)$. We have the following result regarding the set of feasible plans:

LEMMA 3.14: *For any* $x \in \mathcal{X}$ *and* $t \in \mathbb{N}$, $H_t(x, z) \neq \emptyset$.

This result allows us to define a supremum function as

$$v_t^*(x, z) \equiv \sup_{h \in H_t(x,z)} V_t(h, x, z). \tag{27}$$

We first observe that t plays no role in the above equation (27), that is, we prove the following:

LEMMA 3.15: *For any* $t \in \mathbb{N}$ *and* $(x, z) \in \mathcal{X} \times \mathcal{Z}$, $v_t^*(x, z) = v_1^*(x, z)$.

Thus, we are able to drop the subscript t from (27) and write $v^*(x, z)$ instead of $v_t^*(x, z)$.
The next step is to relate v^* to v^τ, the solution of the functional equation studied in the previous subsection, which was proved to exist in Theorem 3.11 and satisfies the Bellman equation:

$$v^\tau(x, z) = \sup_{y \in \Gamma(x,z)} \{u(x, y, z) + \beta \mathrm{Q}_\tau[v^\tau(y, w)|z]\}. \tag{28}$$

In the rest of this section, we will denote v^τ simply by v.
To achieve this aim, we first establish important results relating v in equation (28) to the policy function that solves the original problem. In particular, the next result allows us to define the policy function as follows:

LEMMA 3.16: *If v is a bounded continuous function satisfying* (28), *then for each* $(x, z) \in \mathcal{X} \times \mathcal{Z}$, *the correspondence* $Y : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{X}$ *defined by*

$$Y(x, z) \equiv \{y \in \Gamma(x, z) : v(x, z) = u(x, y, z) + \beta \mathrm{Q}_\tau[v(y, w)|z]\}$$

is nonempty, upper semi-continuous, and has a measurable selection.

Let $\psi : \mathcal{X} \times \mathcal{Z} \to \mathcal{X}$ be a measurable selection of Y. The policy function ψ generates the plan h^ψ defined by $h_t^\psi(z^t) = \psi(h_{t-1}(z^{t-1}), z_t)$ for all $z^t \in \mathcal{Z}^t$, $t \in \mathbb{N}$.
The next result provides sufficient conditions for a solution v to the functional equation to be the supremum function, and for the plans generated by the associated policy function ψ to attain the supremum.

PROPOSITION 3.17: *Let* $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *be bounded and satisfy the functional equation* (28) *and* ψ *be defined as above. Then,* $v = v^*$ *and the plan* h^ψ *attains the supremum in* (27).

3.7. *Euler Equation*

The final step is to characterize the solutions of the problem through the Euler equation. Let $v = v^\tau$ be the unique fixed point of $\mathbb{M}^\tau$, satisfying (28). By Theorem 3.12, v is differentiable in its first coordinate, satisfying $v_{x_i}(x, z) = \frac{\partial v^\tau}{\partial x_i}(x, z) = \frac{\partial u}{\partial x_i}(x, y^*, z) = u_{x_i}(x, y^*, z)$.
Given that we have shown the differentiability of value function, we are able to apply the standard technique to obtain the Euler equation, as formalized in the following theorem:

THEOREM 3.18: *Let Assumption 2 hold and let* $h = h^\psi$ *be an optimal plan, as in Proposition 3.17. Assume that* $x_{t+1}^h \in \operatorname{int} \Gamma(x_t^h, z_t)$, *that is, the optima are interior, and* $z_t \mapsto \frac{\partial u}{\partial x_i}(x_t^h, x_{t+1}^h, z_t)$ *is strictly increasing. Then, the following first-order condition* (*called Euler equation in this setting*) *necessarily holds for every* $t \in \mathbb{N}$ *and* $i = 1, \dots, p$:

$$u_{y_i}(x_t^h, x_{t+1}^h, z_t) + \beta \mathrm{Q}_\tau[u_{x_i}(x_{t+1}^h, x_{t+2}^h, z_{t+1})|z_t] = 0. \tag{29}$$

In the expression above, u_y represents the derivative of u with respect to (some of the coordinates of) its second variable (y) and u_x represents the derivative of u with respect to (some of the coordinates of) its first variable (x).

Theorem 3.18 provides the Euler equation, that is, the optimality conditions for the quantile dynamic programming problem. This result is the generalization of the traditional expected utility to the quantile utility. The Euler equation in (29) is displayed as an implicit function; nevertheless, for any particular application, and given utility function, one is able to solve it explicitly as a conditional quantile function.

The proof of Theorem 3.18 relies on a result about the differentiability inside the quantile function. Indeed, if h is differentiable and the derivative $\frac{\partial h}{\partial x_i}(x, Z)$ is integrable, then

$$\frac{\partial}{\partial x_i}\mathrm{E}\big[h(x, Z)\big] = \mathrm{E}\left[\frac{\partial h}{\partial x_i}(x, Z)\right], \quad \text{but} \quad \frac{\partial}{\partial x_i}\mathrm{Q}_\tau\big[h(x, Z)\big] \neq \mathrm{Q}_\tau\left[\frac{\partial h}{\partial x_i}(x, Z)\right],$$

in general. However, we are able to establish this property under some assumptions. We are not aware of this result in the theory of quantiles, and given its usefulness, we state it here:

PROPOSITION 3.19: *Let $\mathcal{X} \subset \mathbb{R}^p$ be a neighborhood of x, $\mathcal{Z} \subset \mathbb{R}$ is an interval, and assume that $h : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ is continuously differentiable in x and continuous in z, and that h and d are increasing in z, where $d(z) \equiv h(x'_i, x_{-i}, z) - h(x_i, x_{-i}, z)$ for x_i, x'_i satisfying $0 < x'_i - x_i < \epsilon$, for some small $\epsilon > 0$. Then,*

$$\frac{\partial \mathrm{Q}_\tau}{\partial x_i}\big[h(x, Z)\big] = \mathrm{Q}_\tau\left[\frac{\partial h}{\partial x_i}(x, Z)\right].$$

The two conditions in the above result appear frequently in economic models. To see this, consider the following corollary:

COROLLARY 3.20: *Assume that $h : \mathbb{R}^2 \to \mathbb{R}$ is C^2. If*

$$\frac{\partial h(x, z)}{\partial z} \geq 0 \quad \textit{and} \quad \frac{\partial^2 h(x, z)}{\partial x \partial z} \geq 0, \tag{30}$$

then,

$$\frac{\partial \mathrm{Q}_\tau}{\partial x}\big[h(x, Z)\big] = \mathrm{Q}_\tau\left[\frac{\partial h}{\partial x}(x, Z)\right]. \tag{31}$$

One can show that (30) implies that $h(x, z)$ and $d(z) \equiv h(x', z) - h(x, z)$ are increasing in z for $x' > x$. In particular, (30) is easy to check and implies that h is supermodular, which is a property widely used in economics.

To see that (30) is natural in dynamic models, consider the following particular example. In each period t, the consumer decides how much to consume c_t and save for the next period, x_{t+1}, up to the wealth available in period t, which is given by the amount saved in the previous period x_t, multiplied by the shock z_t. That is, the budget equation is $c_t + x_{t+1} \leq x_t z_t$, which leads to $c_t = x_t z_t - x_{t+1}$. If we define $u(x_t, x_{t+1}, z_t) = x_t z_t - x_{t+1}$ and fix y, then $h(x, z) \equiv u(x, y, z) = xz - y$ clearly satifies (30).[29,30]

[29]It is not difficult to see that $h(x, z) \equiv U(xz - y)$ satisfies (30) for any U sufficiently close to a linear function.

[30]In fact, it is useful to illustrate why (31) is valid for this h:

$$\frac{\partial}{\partial x}\mathrm{Q}_\tau\big[h(x, Z)\big] = \frac{\partial}{\partial x}\mathrm{Q}_\tau[xZ - y] = \frac{\partial}{\partial x}\big(x\mathrm{Q}_\tau[Z] - y\big) = \mathrm{Q}_\tau[Z] = \mathrm{Q}_\tau\left[\frac{\partial(xZ - y)}{\partial x}\right] = \mathrm{Q}_\tau\left[\frac{\partial h}{\partial x}(x, Z)\right].$$

4. APPLICATION: INTERTEMPORAL CONSUMPTION MODEL

This section illustrates the usefulness of the new quantile utility maximization methods through an empirical example. We apply the methodology to the standard intertemporal allocation of consumption model, which is central to contemporary economics and finance. It has been used extensively in the literature and has had remarkable success in providing empirical estimates for the study of the elasticity of intertemporal substitution (EIS) and discount-factor parameters. We refer the readers to Campbell (2003), Cochrane (2005), and Ljungqvist and Sargent (2012), and the references therein, for a comprehensive overview.

There is a large empirical literature that attempts to estimate the EIS; among others, Hansen and Singleton (1983), Hall (1988), Campbell and Mankiw (1989), Epstein and Zin (1991), Blundell, Browning, and Meghir (1994), Attanasio and Browning (1995), Atkeson and Ogaki (1996), Campbell and Viceira (1999), and Yogo (2004). The majority of the literature relies on the traditional expected utility framework.

4.1. *Economic Model*

We employ a variation of Lucas's (1978) endowment economy (see, also, Hansen and Singleton (1982)). The economic agent decides on the intertemporal consumption and savings (assets to hold) over an infinite horizon economy, subject to a linear budget constraint. The decision generates an intertemporal policy function, which is used to estimate the parameters of interest for a given utility function.

Let c_t denote the amount of consumption good that the individual consumes in period t. At the beginning of period t, the consumer has x_t units of the risky asset, which pays dividend z_t. The price of the consumption good is normalized to 1, while the price of the risky asset in period t is $p(z_t)$. Then, the consumer decides how many units of the risky asset x_{t+1} to save for the next period, and its consumption c_t.[31] Using the notation introduced in (23), we can write the consumer problem as[32]

$$\max_{(c_t)_{t=0}^{\infty}} \mathrm{Q}_{\tau}^{\infty}\left[\sum_{t=0}^{\infty} \beta_{\tau}^{t} U(c_t)\right], \tag{32}$$

subject to

$$c_t + p(z_t)x_{t+1} \leq \left[z_t + p(z_t)\right] \cdot x_t, \tag{33}$$

$$c_t, x_{t+1} \geq 0, \tag{34}$$

where $\beta_\tau \in (0,1)$ is the discount factor for the quantile $\tau \in (0,1)$ of interest, and $U : \mathbb{R}_+ \to \mathbb{R}$ is the utility function. Note that β_τ is written with a subscript τ, to emphasize the fact that we may have a different parameter for each $\tau \in (0,1)$. Because there is a single agent, the holdings must not exceed one unit. In fact, in equilibrium, we must have $x_{tk}^{*} = 1, \forall t, k$. Let $\bar{x} > 1$ and $\mathcal{X} = [0, \bar{x}]$.

[31] In this model, for simplicity, the state variable next period, x_{t+1}, does not depend directly on the shock next period, z_{t+1}. In such a case, the model resolution is more involved, and a law of motion for the endogenous state variable needs to be defined.

[32] Recall (23): $\mathrm{Q}_{\tau}^{\infty}[\sum_{t=0}^{\infty} \beta_{\tau}^{t} u(c_t)] = u(c_0) + \mathrm{Q}_{\tau}[\beta_{\tau} U(c_1) + \mathrm{Q}_{\tau}[\beta_{\tau}^{2} U(c_2) + \mathrm{Q}_{\tau}[\beta_{\tau}^{3} U(c_3) + \cdots |\Omega_2]|\Omega_1]|\Omega_0]$.

From (33), we can determine the consumption entirely from the current and future states, that is, $c_t = z_t \cdot x_t + p(z_t) \cdot (x_t - x_{t+1})$. Following the notation of the previous sections, we denote x_t by x, x_{t+1} by y, and z_t by z. Then, the above restrictions are captured by the feasible correspondence $\Gamma : \mathcal{X} \times \mathcal{Z} \to \mathcal{X} = \mathcal{X}$ defined by

$$\Gamma(x, z) \equiv \{y \in \mathcal{X} : p(z) \cdot y \leq (z + p(z)) \cdot x\}.$$

For each pricing function $p : \mathcal{Z} \to \mathbb{R}_+$, define the utility function as

$$u(x, y, z) \equiv U[z \cdot x + p(z) \cdot (x - y)].$$

We assume the following:

ASSUMPTION 3: (i) *$\mathcal{Z} \subseteq \mathbb{R}_{++}$ is a bounded interval and $\mathcal{X} = [0, \bar{x}]$;*
(ii) *$U : \mathbb{R}_+ \to \mathbb{R}$ is given by $U(c) = \frac{1}{1-\gamma} c^{1-\gamma}$, for $\gamma > 0$, $\gamma \neq 1$;*
(iii) *z follows a Markov process with pdf $f : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$, which is continuous, symmetric, $f(z, w) > 0$, for all $(z, w) \in \mathcal{Z} \times \mathcal{Z}$ and satisfies the property: if $h : \mathcal{Z} \to \mathbb{R}$ is weakly increasing and $z \leq z'$, then*

$$\int_{\mathcal{Z}} h(\alpha) f(\alpha|z)\, d\alpha \leq \int_{\mathcal{Z}} h(\alpha) f(\alpha|z')\, d\alpha;$$

(iv) *$z \mapsto z + p(z)$ is C^1 and non-decreasing, with $z(\ln(z + p(z)))' > \gamma$.*

Assumptions 3(i)–(ii) are standard in economic applications. Assumption 3(ii) specifies an isoelastic utility function (constant elasticity of substitution—CES). This is a standard assumption in a large body of the literature, as for example, among others, Hansen and Singleton (1982), Stock and Wright (2000), and Campbell (2003). Assumption 3(iii) states that the idiosyncratic shocks follow a Markov process and that a high value of the dividend today makes a high value tomorrow more likely. It implies Assumption 2(ii). Assumption 3(iv), $z \mapsto z + p(z)$ is non-decreasing, is natural. It states that the price of the risky asset and its return are a non-decreasing function of the dividends. Note that it is natural to expect that the price is non-decreasing with the dividends, but Assumption 3(iv) is even weaker than this, as it allows the price to decrease with the dividend; only $z + p(z)$ is required to be non-decreasing.[33]

Given Assumption 3, we can verify the assumptions for establishing the quantile utility model in the intertemporal consumption model context. Thus, we have the following:

LEMMA 4.1: *Assumption 3 implies Assumptions 1 and 2 and that $z_t \mapsto \frac{\partial u}{\partial x_i}(x_t^h, x_{t+1}^h, z_t)$ is strictly increasing.*

Therefore, Theorems 3.11 and 3.12 imply the existence of a value function v, which is strictly concave and differentiable in its first variable, satisfying

$$v(x, z) = \max_{y \in \Gamma(x,z)} \{U[z \cdot x + p(z) \cdot (x - y)] + \beta_\tau \mathrm{Q}_\tau[v(y, z')|z]\},$$

[33] In our data set, when regressing the returns on the dividends, we find a statistically positive correlation.

where $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}$. Note that

$$\frac{\partial u}{\partial x}(x, y, z) = U'\big[z \cdot x + p(z) \cdot (x - y)\big]\big(z + p(z)\big);$$
$$\frac{\partial u}{\partial y}(x, y, z) = U'\big[z \cdot x + p(z) \cdot (x - y)\big]\big(-p(z)\big).$$

Because, in equilibrium, the holdings are $x_t = 1$ for all t, we can derive the Euler equation as in (29) for this particular problem to obtain

$$-p(z_t)U'(c_t) + \beta_\tau \mathrm{Q}_\tau\big[U'(c_{t+1})\big(z_{t+1} + p(z_{t+1})\big)|\Omega_t\big] = 0. \tag{35}$$

Let us define the return by $1 + r_{t+1} \equiv \frac{z_{t+1}+p(z_{t+1})}{p(z_t)}$. Assumption 3(i) implies that the ratio of marginal utilities can be written as $\frac{U'(c_{t+1})}{U'(c_t)} = (\frac{c_{t+1}}{c_t})^{-\gamma}$. Thus, the Euler equation in (35) simplifies to

$$\mathrm{Q}_\tau\left[\beta_\tau(1 + r_{t+1})\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma_\tau} - 1\middle|\Omega_t\right] = 0. \tag{36}$$

Now, for illustration purposes, we compare the quantile utility maximization results with the corresponding ones for the expected utility, which can be written as

$$\max_{(c_t)_{t=0}^{\infty}} \mathrm{E}\left[\sum_{t=0}^{\infty} \beta^t U(c_t)\right],$$

subject to the same constraints in (33)–(34). This problem can be rewritten and the associated value function is

$$v(x, z) = \max_{y\in\Gamma(x,z)} \big\{U\big[z \cdot x + p(z) \cdot (x - y)\big] + \beta\mathrm{E}\big[v\big(y, z'\big)|z\big]\big\}.$$

Finally, the corresponding Euler equation can be expressed as

$$-p(z_t)U'(c_t) + \beta\mathrm{E}\big[U'(c_{t+1})\big(z_{t+1} + p(z_{t+1})\big)|\Omega_t\big] = 0,$$

and by rearranging the previous equation, we obtain

$$\mathrm{E}\left[\beta(1 + r_{t+1})\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} - 1\middle|\Omega_t\right] = 0. \tag{37}$$

When comparing the Euler equations in (36) and (37), one can notice similarities and differences. The expressions inside the conditional quantile in (36) and the conditional expectation in (37) are practically the same, except that, for the quantile model, the parameters $(\beta_\tau, \gamma_\tau)$ depend on the quantile τ. That is, for each τ, we may have (potentially) different parameters.

4.2. *Estimation*

The previous section derives the Euler equation for the quantile utility model. Now we describe how to estimate the vector of parameters of interest. The basic idea underlying the estimation strategy is to use the theoretical economic model to generate a family of nonlinear conditional quantile functions, and apply the method of moments instrumental variables (IV) quantile regression (QR) for nonlinear models developed in de Castro, Galvao, Kaplan, and Liu (2019).[34]

For a given parameterized utility function, one is able to isolate the implicit quantile function defined by equation (29). In particular, (36) can be written as the following nonlinear conditional quantile model:

$$\mathrm{Q}_\tau\big[m(c_t, r_t, \theta_{0\tau})|\Omega_t\big] = 0, \tag{38}$$

where $\tau \in (0, 1)$ is a quantile of interest, $m(c_t, r_t, \theta_{0\tau}) \equiv \beta_\tau(1 + r_{t+1})(\frac{c_{t+1}}{c_t})^{-\gamma_\tau} - 1$, with $\theta_{0\tau} = (\beta_\tau, \gamma_\tau)^\top \in \mathcal{B} \subseteq \mathbb{R}^2$, and Ω_t denoting the σ-field that contains the information up to time t. The vector (c_t, r_t, w_t) contains the observable variables, with consumption $c_t \in \mathcal{Y}$, the real return on the asset $r_t \in \mathcal{X}$, the full instrument vector $w_t \in \mathcal{W}$. The quantile model in (38) is valid for each given quantile τ. We aim to estimate the parameters $\theta_{0\tau}$ that describe the Euler equation for specified quantiles of interest.

The model in (38) can be represented by non-smooth conditional moment restrictions as

$$\mathrm{E}\big[\tau - 1\big\{m(c_t, r_t, \theta_{0\tau}) \le 0\big\}|w_t\big] = 0, \tag{39}$$

where $1\{\cdot\}$ is the indicator function. Since $\mathrm{E}[1\{m(c_t, r_t, \theta_{0\tau}) \le 0\}|w_t] = F[m(c_t, r_t, \theta_{0\tau})|w_t]$, when $F[\cdot]$ is invertible, one is able to recover (38) from (39).

For a given quantile index τ, estimation of the parameter vector $\theta_{0\tau}$ uses the method of moments. Rewrite (39) as the following moment condition:

$$\mathrm{E}\big[w_t\big[1\big\{m(c_t, r_t, \theta_{0\tau}) \le 0\big\} - \tau\big]\big] = 0, \tag{40}$$

where instruments, w_t, are used to achieve a valid orthogonality condition.

We now present the smoothed instrumental variables quantile regression (SIVQR) estimator. Let the population map $M : \mathcal{B} \times \mathcal{T} \mapsto \mathbb{R}^2$ be

$$M(\theta, \tau) \equiv \mathrm{E}\big[g_t^u(\theta, \tau)\big], \tag{41}$$

$$g_t^u(\theta, \tau) \equiv g^u(c_t, r_t, w_t, \theta, \tau) \equiv w_t\big[1\big\{m(c_t, r_t, \theta) \le 0\big\} - \tau\big], \tag{42}$$

where superscript "u" denotes "unsmoothed." The population moment condition (40) is

$$M(\theta_{0\tau}, \tau) = 0. \tag{43}$$

The method of moments is constructed by replacing the population moments, the expectation $\mathrm{E}[\cdot]$, with their corresponding sample expectation $\widehat{\mathrm{E}}[\cdot]$, that is, the sample average. Analogously to (41), using (42), the unsmoothed *sample* moment map is

$$\widehat{M}_T^u(\theta, \tau) \equiv \widehat{\mathrm{E}}\big[g^u(c, r, w, \theta, \tau)\big] \equiv \frac{1}{T}\sum_{t=1}^{T} g_t^u(\theta, \tau). \tag{44}$$

[34]It has been standard in the literature to estimate Euler equations derived from the expected utility models. It is an important exercise to learn about the structural parameters that characterize the economic problem of interest. After parameterizing the utility function, the restrictions imply a conditional average model and the parameters are commonly estimated by the generalized method of moments (GMM) of Hansen (1982). Estimation and inference of GMM have been discussed by, among many others, Newey and McFadden (1994), Chen, Linton, and van Keilegom (2003), Chen and Pouzo (2009), and Chen and Liao (2015).

Even if population system (43) has a unique solution, the unsmoothed system $\widehat{M}_T^u(\theta, \tau) = 0$ may have zero or multiple solutions. Although this issue can be overcome theoretically, smoothing addresses it directly. The SIVQR estimator is the solution to the system of smoothed sample moments. With smoothing (no "u" superscript), the sample analogs of (41), (42), and (43) are

$$\begin{aligned} g_{Tt}(\theta, \tau) &\equiv g_T(c_t, r_t, w_t, \theta, \tau) \equiv w_t\big[\tilde{I}\big(-m(c_t, r_t, \theta)/h_T\big) - \tau\big], \\ \widehat{M}_T(\theta, \tau) &\equiv \frac{1}{T}\sum_{t=1}^{T} g_{Tt}(\theta, \tau), \end{aligned} \tag{45}$$

$$\widehat{M}_T(\hat{\theta}_\tau, \tau) = 0, \tag{46}$$

where h_T is a bandwidth (sequence), $\tilde{I}(\cdot)$ is a smoothed version of the indicator function $1\{\cdot > 0\}$, and $I(\cdot)$ may stand for "indicator-like function" or "integral of kernel." An example function $\tilde{I}(\cdot)$ is the integral of a fourth-order polynomial kernel.

Finally, given a random sample $\{(c_t, r_t, w_t) : t = 1, \dots, T\}$, the parameters $\theta_{0\tau}$ can be estimated by (45) and (46). The objective function depends only on the available sample information, the known function $m(\cdot)$, and the unknown parameters. Solutions of the above problem are denoted by $\widehat{\theta}_\tau$, the SIVQR estimator. de Castro et al. (2019) discussed and gave conditions for identification of the parameters of interest, and considered estimation and inference with weakly dependent data. The parameter $\theta_{0\tau}$ is "locally identified" if there exists a neighborhood of $\theta_{0\tau}$ in which only $\theta_{0\tau}$ satisfies (40). This property holds if the partial derivative matrix of the right-hand side of (40) with respect to the θ argument is full rank.[35] In addition, de Castro et al. (2019) established the asymptotic properties of the SIVQR estimator.

PROPOSITION 4.2—de Castro et al. (2019): *Under standard regularity conditions, as* $T \to \infty$, *the estimator is consistent, that is,* $\widehat{\theta}_\tau \xrightarrow{p} \theta_{0\tau}$, *and*

$$\sqrt{T}(\widehat{\theta}_\tau - \theta_{0\tau}) \xrightarrow{d} N\big(0, G^{-1}\Sigma_{M_\tau}\big[G^\top\big]^{-1}\big),$$

where $\Sigma_{M_\tau} = \lim_{T\to\infty} \operatorname{Var}(T^{-1/2}\sum_{t=1}^{T} g_{Tt}(\theta_{0\tau}, \tau))$, $G = \frac{\partial}{\partial\theta^\top}\mathrm{E}[w_t 1\{m(c_t, r_t, w_t, \theta) \leq 0\}]|_{\theta=\theta_{0\tau}} = -\mathrm{E}[w_t d_t^\top f(0|w_t, d_t)]$, $d_t \equiv \nabla_\theta m(c_t, r_t, \theta_{0\tau})$, *and* $f_{m|w,d}(\cdot|w, d)$ *is the conditional pdf of* m_t *given* $w_t = w$ *and* $d_t = d$.

Given the result in Proposition 4.2, one is able to estimate the variance-covariance matrix and conduct practical inference. One is also able to apply the SIVQR methods and estimate $(\gamma_\tau, \beta_\tau)$ across different quantiles by simply varying τ. One advantage of the quantile Euler equation is that it may be log-linearized with no approximation error, differently from the standard Euler equation. Thus, we use a model as $\mathrm{Q}_\tau[-\ln(c_{t+1}/c_t) + \gamma_\tau^{-1}\ln(\beta_\tau) + \gamma_\tau^{-1}\ln(1 + r_{t+1})|\Omega_t] = 0$. From $\ln(\beta_\tau)/\gamma_\tau$ and $1/\gamma_\tau$, we are able to recover the parameters of interest.

REMARK 4.3: In this paper, we are interested in estimating the conditional quantile functions to learn about the potential underlying heterogeneity among quantiles. Nevertheless, it is possible to see the quantile τ as a parameter to be estimated together with

[35]Identification for general nonlinear semiparametric and nonparametric conditional moment restrictions models is presented in Chen et al. (2014).

the parameters $\theta_{0\tau}$. Bera, Galvao, Montes-Rojas, and Park (2016) developed an approach that delivers estimates for the coefficients together with a representative quantile.In their framework, τ captures a measure of asymmetry of the conditional distribution of interest and is associated with the "most probable" quantile in the sense that it maximizes the entropy.

4.3. *Data*

We use a data set that is common in the literature. We use monthly data from 1959:01 to 2015:11, which produces 683 observations. As is standard in the literature (see, e.g., Hansen and Singleton (1982)), two different measures of consumption were considered: nondurables, and nondurables plus services. The monthly, seasonally adjusted observations of aggregate nominal consumption (measured in billions of dollars unit) of nondurables and services were obtained from the Federal Reserve Economic Data. Real per capita consumption series were constructed by dividing each observation of these series by the corresponding observation of population, deflated by the corresponding CPI (base 1973:01).

Each measure of consumption was paired with four sets of stock returns from the Center for Research in Security Prices (CRSP) U.S. Stock database, which contains month-end prices for primary listings for the New York Stock Exchange (NYSE). We use the equally-weighted average of returns (EWR) (including and excluding dividends) on the NYSE. The nominal returns were converted to real returns by dividing by the deflator associated with the measure of consumptions. Instruments include lags of log real consumption growth, nominal interest rate, and inflation. We use two instruments (similar to the excluded instruments used in Yogo (2004)): the linear projection of the real interest rate and log consumption growth onto a constant and nominal interest rate, inflation, and lagged log consumption growth. All instruments are lagged twice to avoid problems with time aggregation in consumption data.

4.4. *Results*

Before we present the estimation results, it is important to discuss the interpretation of the parameters of interest $(\beta_\tau, 1/\gamma_\tau)$. We notice that, as discussed in Section 2.3, it is possible to separate the risk attitude from the intertemporal substitutability in the quantile model. First, the present notion of risk preference differs in several respects from the one familiar in the expected utility literature. The quantile τ captures the risk attitude in the model. Given that the notion of risk attitude is comparative and captured by varying the quantile index, we estimate the model for several different quantiles. Thus, an important point in the application is to compare estimates across quantiles, that is, different measures of risk. Second, for a given quantile τ, β_τ is the usual discount factor. Finally, from the discussion in Section 2.3 and equation (32), one can notice that the parameter $1/\gamma_\tau$ captures the standard measure of EIS implicit in the CES utility function. Thus, by employing the quantile maximization model, for each specific risk attitude τ, we are able to estimate the associated discount factor and EIS.

Now we present the empirical results. For comparison purposes, we also report results for the standard conditional average instrumental variables regression by two stage least squares (TSLS), which estimates the parameters for the expected utility model. The results for the estimates of the parameters at different quantiles are reported in Table I and Figure 3. We present estimates using consumption of nondurables and stock return

TABLE I

SIVQR AND TSLS ESTIMATES FOR DISCOUNT FACTOR AND EIS[a]

	EWRwo		EWRw	
τ	β_τ	$1/\gamma_\tau$	β_τ	$1/\gamma_\tau$
0.1	1.156	0.105	1.147	0.110
	(0.21)	(0.02)	(0.19)	(0.02)
0.2	1.061	0.165	1.059	0.158
	(0.03)	(0.02)	(0.20)	(0.09)
0.3	1.033	0.190	1.028	0.194
	(0.02)	(0.03)	(0.05)	(0.07)
0.4	1.006	0.380	1.003	0.366
	(0.01)	(0.18)	(0.01)	(0.17)
0.5	0.991	0.314	0.989	0.350
	(0.01)	(0.04)	(0.01)	(0.10)
0.6	0.979	0.543	0.974	0.199
	(0.01)	(0.27)	(0.04)	(0.07)
0.7	0.968	0.836	0.963	0.505
	(0.02)	(1.12)	(0.04)	(0.41)
0.8	0.802	0.159	0.844	0.183
	(0.53)	(0.07)	(0.91)	(0.19)
0.9	0.806	0.300	0.725	0.159
	(0.13)	(0.09)	(0.46)	(0.05)
TSLS	0.992	0.203	0.989	0.204
	(0.03)	(0.19)	(0.04)	(0.18)

[a] This table shows coefficients returned from applying SIVQR and TSLS methods to estimate the Euler equation. Standard errors in parentheses. The smoothing bandwidth is $h = 10^{-4}$ for EWRwo and $h = 10^{-3}$ for EWRw.

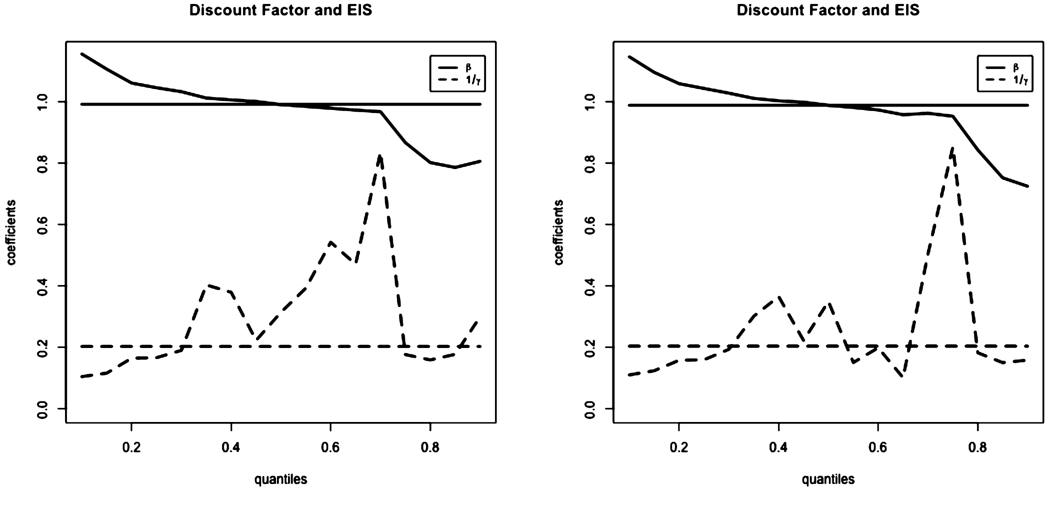


FIGURE 3.—Discount factor and EIS estimates for SIVQR and TSLS using nondurables plus services for EWRwo (left panel) and EWRw (right panel). Straight lines correspond to TSLS, while SIVQR appear as curves.

with and without dividends. The panels on the left display the estimates for EWR excluding dividends (EWRwo), and the right panel including dividends (EWRw). The results for consumption nondurables plus services are qualitatively similar; for brevity, we omit them.

First, we consider the estimates of the discount factor using EWRwo. The results show empirical evidence that estimates of the discount factor are decreasing across quantiles. For the upper quantiles, the estimates are close to 0.80. Table I shows that, for low quantiles, the discount factor estimates are larger than 1. Nevertheless, Epstein and Zin (1991, p. 282) also estimated a discount factor larger than 1. Overall, Table I shows evidence that the discount factor is relatively smaller for upper quantiles (more risk preferring). The results for the TSLS case are consistent with the literature and show a discount factor of 0.992.

Next, we consider the estimates of the EIS, $1/\gamma_\tau$. The left panels in Table I and Figure 3 present the results using EWRwo. The first interesting observation is that the results document evidence of heterogeneity in EIS across quantiles, while the TSLS provides an estimate of 0.203. In particular, the table shows that, overall, the EIS increases across quantiles, especially for $\tau \in (0.1, 0.7)$, such that EIS is relatively larger for upper quantiles. The smaller EIS, for low quantiles (less risk preferring), means less sensitivity to changes in intertemporal consumption. There is an existing literature exploring whether the EIS varies with the level of consumption (or wealth) which rejects the constant-EIS hypothesis (Blundell, Browning, and Meghir (1994); Attanasio and Browning (1995); Atkeson and Ogaki (1996)). In this illustration, we shed light on the discussion and allow the EIS to vary with the risk attitude, indexed by the quantile.

The right panels in Table I and Figure 3 display the estimates when considering EWRw. They serve as a robustness check. The results are qualitatively similar to those in the left panel and Figure 3. The coefficients of EIS present variation over the quantiles. The discount factor estimates also present heterogeneity, especially for upper quantiles. The discount factor is smaller for larger quantiles (more risk taking).

REMARK 4.4: It should be noted that our model does not control for income or wealth. Thus, the agents that maximize low quantiles *do not* necessarily correspond to low income. Instead, those agents dislike risk more than agents that maximize high quantiles. This observation is important to avoid confusion with the results in a branch of the literature that links discount factors with income and wealth (see, e.g., Hausman (1979), and Lawrance (1991)). Moreover, there is empirical evidence that documents small discount factors estimates. This literature estimates discount factors by using a quasi-hyperbolic discount function (see, e.g., Paserman (2008), and Fang and Silverman (2009)). In contrast to these streams of literature, this paper abstracts from a relationship between discount rates and poverty and employs a simple model to estimate the discount factor. Our objective is to illustrate the potential empirical application of the quantile utility maximization model. We leave the connection with income and wealth and related extensions for future research.

In all, the application illustrates that the new methods serve as an important alternative tool to study economic behavior. The methods allow one to estimate the discount factor and EIS at different levels of risk attitude (quantiles). Our empirical results document heterogeneity in both discount factor and EIS across quantiles.

5. SUMMARY AND OPEN QUESTIONS

This paper develops a dynamic model of rational behavior under uncertainty for an agent maximizing the quantile utility function indexed by a quantile $\tau \in (0,1)$.[36] More specifically, an agent maximizes the stream of future τ-quantile utilities, where the quantile preferences induce the quantile utility function. We show dynamic consistency of the recursive quantile preferences and that this dynamic problem yields a value function, using a fixed point argument. We also obtain desirable properties of the value function. In addition, we derive the corresponding Euler equation.

Empirically, we show that one can employ existing general instrumental variables nonlinear quantile regression methods for estimating and testing the rational quantile models directly from stochastic Euler equations. An attractive feature of this method is that the parameters of the dynamic objective functions of economic agents can be interpreted as structural objects. Finally, to illustrate the methods, we construct an intertemporal consumption model and estimate the implied discount factor and elasticity of intertemporal substitution (EIS) parameters for different quantiles. The results provide evidence that both discount factor and EIS vary across quantiles.

Many issues remain to be investigated. An interesting direction for future study would be to generalize our model to the case where the future state is randomly defined instead of directly chosen. Extensions of the methods to general equilibrium models pose challenging new questions as how to endogenize prices, balance supply and demand, and describe an efficient allocation of goods and services in the economy. In addition, aggregation of the quantile preferences is also a critical direction for future research. Another important avenue is to investigate the relationship between the quantile utility model and more general rank-dependent models of choice under uncertainty proposed by Quiggin (1982) and Yaari (1987). Applications to asset pricing, consumption models, and scenario-based analysis would appear to be a natural direction for further development of quantile utility maximization models.

APPENDIX

Given a random variable (r.v.) X, let F_X (or simply F) denote its cumulative distribution function (c.d.f.), that is, $F_X(\alpha) \equiv \Pr[X \leq \alpha]$. If X is clear from the context and we can omit it, the quantile function $Q : [0,1] \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is the generalized inverse of F:

$$Q(\tau) \equiv \begin{cases} \inf\{\alpha \in \mathbb{R} : F(\alpha) \geq \tau\} & \text{if } \tau \in (0,1], \\ \sup\{\alpha \in \mathbb{R} : F(\alpha) = 0\} & \text{if } \tau = 0. \end{cases}$$

The definition is special for $\tau = 0$ so that the quantile assumes a value in the support of X.[37] It is clear that if F is invertible, that is, if F is strictly increasing, its generalized inverse coincides with the inverse, that is, $Q(\tau) = F^{-1}(\tau)$. Usually, it will be important to highlight the random variable to which the quantile refers. In this case, we will denote $Q(\tau)$ by $Q_\tau[X]$. For convenience, throughout the paper, we will focus on $\tau \in (0,1)$, unless explicitly stated.

[36] The extension of the results to the cases $\tau = 0$ and $\tau = 1$ are left for future research.

[37] Indeed, $\inf\{\alpha \in \mathbb{R} : F(\alpha) \geq 0\} = -\infty$, no matter what is the distribution.

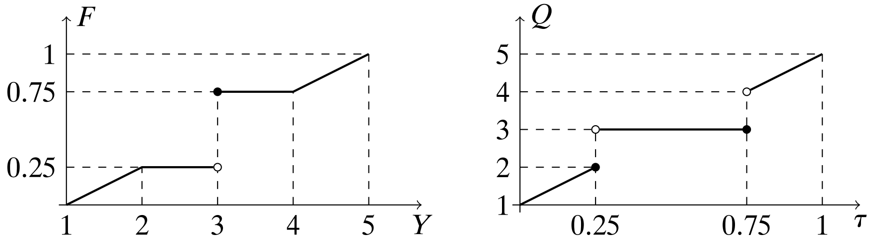


FIGURE 4.—c.d.f. and quantile function of a random variable.

A.1. *Properties of Quantiles*

Figure 4 illustrates the c.d.f. F of a random variable X, and its corresponding quantile function $Q(\tau) = \inf\{\alpha \in \mathbb{R} : F(\alpha) \geq \tau\}$, for $\tau > 0$.[38] In this case, X assumes the value 3 with 50% probability and is uniform in $[1, 2] \cup [4, 5]$ with 50% probability. This picture is useful to inspire some of the properties that we state below. Note, for instance, the discontinuities and the values over which the quantile is constant.

The following lemma is an auxiliary result that will be helpful for the derivations below.

LEMMA A.1: *The following statements are true*:

(i) *Q is increasing*, *that is*, $\tau \leq \hat{\tau} \implies Q(\tau) \leq Q(\hat{\tau})$.

(ii) $\lim_{\tau \downarrow \hat{\tau}} Q(\tau) \geq Q(\hat{\tau})$.

(iii) *Q is left-continuous*, *that is*, $\lim_{\tau \uparrow \hat{\tau}} Q(\tau) = Q(\hat{\tau})$.

(iv) $\Pr[\{z : z < Q(\tau)\}] \leq \tau \leq \Pr[\{z : z \leq Q(\tau)\}] = F[Q(\tau)]$.

(v) *If* $g : \mathbb{R} \to \mathbb{R}$ *is a continuous and strictly increasing function*, *then* $\mathrm{Q}_\tau[g(X)] = g(\mathrm{Q}_\tau[X])$.

(vi) *If* $g, h : \mathbb{R} \to \mathbb{R}$ *are such that* $g(\alpha) \leq h(\alpha)$, $\forall \alpha$, *then* $\mathrm{Q}_\tau[g(Z)] \leq \mathrm{Q}_\tau[h(Z)]$.

(vii) *F is continuous if and only if Q is strictly increasing.*

(viii) *F is strictly increasing if and only if Q is continuous*.

PROOF: (i) Let us first assume $\tau > 0$. If $\tau \leq \hat{\tau}$, then $\{\alpha \in \mathbb{R} : F_Z(\alpha) \geq \tau\} \supseteq \{\alpha \in \mathbb{R} : F_Z(\alpha) \geq \hat{\tau}\}$. This implies $Q_Z(\tau) \leq Q_Z(\hat{\tau})$. Next, if $\sup\{\alpha \in \mathbb{R} : F_Z(\alpha) = 0\} = -\infty$, there is nothing else to prove. If $\sup\{\alpha \in \mathbb{R} : F_Z(\alpha) = 0\} = x \in \mathbb{R}$, then $F_Z(x - \epsilon) = 0$ for any $\epsilon > 0$. Let $\hat{\tau} > 0$. Then, $y \in \{\alpha \in \mathbb{R} : F_Z(\alpha) \geq \hat{\tau}\} \implies y > x - \epsilon$, which in turn implies $Q_Z(\hat{\tau}) \geq x - \epsilon$. Since $\epsilon > 0$ is arbitrary, this implies $Q_Z(\hat{\tau}) \geq x = Q_Z(0)$, which concludes the proof.

(ii) From (i), $\lim_{\tau \downarrow \hat{\tau}} Q_Z(\tau) \geq \inf_{\tau \geq \hat{\tau}} Q_z(\tau) \geq Q_z(\hat{\tau})$. Figure 4 illustrates (e.g., for $\hat{\tau} = 0.25$) that the inequality can be strict.

(iii) From (i), we know that $\lim_{\tau \uparrow \hat{\tau}} Q_Z(\tau) \leq Q_z(\hat{\tau})$. For the other inequality, assume that $\lim_{\tau \uparrow \hat{\tau}} Q_Z(\tau) + 2\epsilon < Q_z(\hat{\tau}) < \infty$, for some $\epsilon > 0$. This means that for each $k \in \mathbb{N}$, we can find $\alpha^k \in \{\alpha : F_Z(\alpha) \geq \hat{\tau} - \frac{1}{k}\}$ such that $Q_Z(\hat{\tau} - \frac{1}{k}) \leq \alpha^k \leq Q_Z(\hat{\tau} - \frac{1}{k}) + \epsilon < Q_Z(\hat{\tau}) - \epsilon$. We may assume that $\{\alpha^k\}$ is an increasing sequence bounded by $Q_z(\hat{\tau})$ and thus converges to some $\bar{\alpha} \in \mathbb{R}$. Then, $\lim_{\tau \uparrow \hat{\tau}} Q_Z(\tau) \leq \bar{\alpha} \leq Q_z(\hat{\tau}) - \epsilon < Q_z(\hat{\tau})$. Since $F_Z(\alpha^k) \geq \hat{\tau} - \frac{1}{k}$ and F_Z is upper semi-continuous, $F_Z(\bar{\alpha}) \geq \hat{\tau}$, which implies that $\bar{\alpha} \geq Q_Z(\hat{\tau})$, a contradiction. Now, assume that $Q_Z(\hat{\tau}) = \infty$. Since $\lim_{\alpha \to \infty} F_Z(\alpha) = 1$, the set $\{\alpha \in \mathbb{R} : F_Z(\alpha) \geq \tau\}$ is non-empty for all $\tau < 1$, that is, $Q_Z(\tau) < \infty$ for all $\tau < 1$. Thus, $\hat{\tau} = 1$. If $\lim_{\tau \uparrow 1} Q_Z(\tau) = x \in \mathbb{R}$,

[38]For $\tau = 0$, $Q(0) = \sup\{\alpha \in \mathbb{R} : F(\alpha) = 0\}$ is just the lower limit of the support of the variable.

then $F_Z(x+1) \geq 1-\epsilon$ for all $\epsilon > 0$, which implies that $F_Z(x+1) = 1$ and $Q_Z(1) \leq x+1$, a contradiction.

(iv) As above, if $Q_Z(\tau) = \infty$, then $\tau = 1$, which implies $1 = \Pr[\{w : z < \infty\}] = \Pr[\{w : z \leq \infty\}]$ and there is nothing to prove. Let $\bar{\alpha} = Q_Z(\tau) < \infty$. If $\alpha^k \downarrow \bar{\alpha}$ is such that $F_Z(\alpha^k) \geq \tau$, then $F_Z(\bar{\alpha}) \geq \tau$, by the well-known upper-semicontinuity of F_Z. That is, $\tau \leq F_Z[Q_Z(\tau)]$. For the other inequality, let $\alpha^k \uparrow \bar{\alpha} = Q_Z(\tau)$. Since $\alpha^k < \bar{\alpha}$, then $\Pr[Z \leq \alpha^k] < \tau$, by the definition of $\bar{\alpha}$. Thus, $\Pr[Z < \alpha^k] \leq \Pr[Z \leq \alpha^k] < \tau$ and $\Pr[Z < \bar{\alpha}] \leq \sup_k \Pr[Z < \alpha^k] \leq \tau$.

(v) The proof is direct as follows:

$$\begin{aligned}
\mathrm{Q}_\tau\big[g(Z)\big] &= \inf\big\{\alpha \in \mathbb{R} : \Pr\big[g(Z) \leq \alpha\big] \geq \tau\big\} \\
&= \inf\big\{\alpha \in \mathbb{R} : \Pr\big[Z \leq g^{-1}(\alpha)\big] \geq \tau\big\} \\
&= \inf\big\{\alpha \in \mathbb{R} : g^{-1}(\alpha) = \beta, \Pr[Z \leq \beta] \geq \tau\big\} \\
&= \inf\big\{g(\beta) : \Pr[Z \leq \beta] \geq \tau\big\} \\
&= g\big(\inf\big\{\beta : \Pr[Z \leq \beta] \geq \tau\big\}\big) \\
&= g\big(\mathrm{Q}_\tau[Z]\big).
\end{aligned}$$

(vi) Since $g \leq h$, then for any α, $\{z : g(z) \leq \alpha\} \supseteq \{z : h(z) \leq \alpha\}$, which implies $F_{g(Z)}(\alpha) = \Pr[g(Z) \leq \alpha] \geq \Pr[h(Z) \leq \alpha] = F_{h(Z)}(\alpha)$. If $\tau > 0$, $\{\alpha \in \mathbb{R} : \Pr[g(Z) \leq \alpha] \geq \tau\} \supseteq \{\alpha \in \mathbb{R} : \Pr[h(Z) \leq \alpha] \geq \hat{\tau}\}$. Taking infima, we obtain $Q_{g(Z)}(\tau) \leq Q_{h(Z)}(\tau)$. On the other hand, $\{\alpha \in \mathbb{R} : F_{h(Z)}(\alpha) = 0\} \subset \{\alpha \in \mathbb{R} : F_{g(Z)}(\alpha) = 0\}$ and taking the supremum in both sides, we obtain the same conclusion.

(vii) Assume that F_Z is discontinuous at x_0, that is, $\lim_{x\uparrow x_0} F_Z(x) = y_0 < y_1 = F_Z(x_0)$. If $y_0 < y_2 < y_3 < y_1$, then $Q_Z(y_2) = \inf\{\alpha : F_Z(\alpha) \geq y_2\} = \inf\{\alpha : F_Z(\alpha) \geq y_3\} = Q_Z(y_3)$, that is, Q_Z is not strictly increasing. Conversely, assume that Q_Z is not strictly increasing, that is, there exists $y_2 < y_3$ such that $Q_Z(y_2) = Q_Z(y_3) = x$. By definition, this means that $F_Z(x-\epsilon) < y_2 < y_3 \leq F_Z(x+\epsilon)$, for all $\epsilon > 0$. But this implies that F_Z is not continuous at x.

(viii) Suppose that F_Z is not strictly increasing, that is, there exists $x_1 < x_2$ such that $F_Z(x_1) = F_Z(x_2) = y$. Then, $Q_Z(y-\epsilon) = \inf\{\alpha : F_Z(\alpha) \geq y-\epsilon\} \leq x_1 < x_2 \leq \inf\{\alpha : F_Z(\alpha) \geq y+\epsilon\} = Q_Z(y+\epsilon)$. Thus, Q_Z cannot be continuous at y. Conversely, assume that Q_Z is not continuous at y_0. Since Q_Z is increasing by (i) and left-continuous by (iii), this means that $Q_Z(y_0) = x_0 < x_1 = \lim_{y\downarrow y_0} Q_Z(y)$. If $x_0 < x_2 < x_1$, then $F_Z(x_2) \leq y_0$, otherwise $\lim_{y\downarrow y_0} Q_Z(y) \leq x_2$. By (iv), we have $y_0 \leq F_Z(Q_Z(y_0)) = F_Z(x_0) \leq F_Z(x_2) \leq y_0$, that is, F_Z is not strictly increasing between x_0 and x_2. *Q.E.D.*

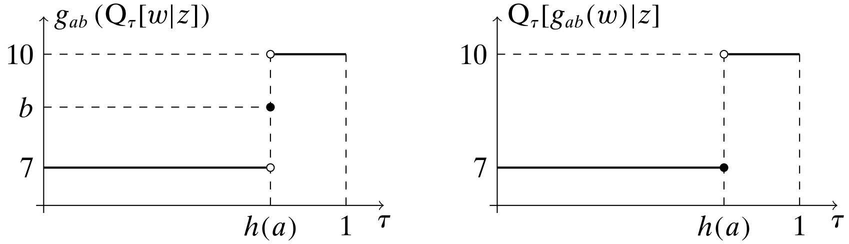
Let Θ be a set (of parameters) and $g : \Theta \times \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ be a measurable function. We denote by $\mathrm{Q}_\tau[g(\theta, \cdot)|z]$ the quantile function associated with g, that is,

$$\mathrm{Q}_\tau\big[g(\theta, \cdot)|z\big] \equiv \inf\big\{\alpha \in \mathbb{R} : \Pr\big[\big(g(\theta, W) \leq \alpha\big)|Z = z\big] \geq \tau\big\}. \tag{47}$$

The following lemma generalizes equation (1) to conditional quantiles.

LEMMA A.2: *Let $g : \Theta \times \mathcal{Z} \to \mathbb{R}$ be non-decreasing and left-continuous in $\mathcal{Z}$. Then,*

$$\mathrm{Q}_\tau\big[g(\theta, \cdot)|z\big] = g\big(\theta, \mathrm{Q}_\tau[w|z]\big). \tag{48}$$



(a): $g_{ab}(\mathrm{Q}_\tau[w|z])$. (b): $\mathrm{Q}_\tau[g_{ab}(w)|z]$.

FIGURE 5.—c.d.f. and quantile function of the random variable g_{ab}.

It is useful to illustrate the above result with an example. Let us define the function $g_{ab} : [1, 5] \to \mathbb{R}$ by

$$g_{ab}(x) = \begin{cases} 7 & \text{if } x < a, \\ b & \text{if } x = a, \\ 10 & \text{if } x > a. \end{cases}$$

The function g_{ab} thus defined is non-decreasing if $b \in [7, 10]$ and it is left-continuous if $b = 7$.

Consider the r.v. X whose c.d.f. F and quantile function Q are shown in Figure 5 above. Let F_{ab} and Q_{ab} denote respectively the c.d.f. and quantile functions associated to $g_{ab}(Z)$. Figure 5 shows $\mathrm{Q}_\tau[g_{ab}(w)|z]$ and $g_{ab}(\mathrm{Q}_\tau[w|z])$ for $a \in [1, 5]$ and $b \in [7, 10]$. The point of discontinuity is a function of a ($h(a) \in [0, 1]$).

PROOF OF LEMMA A.2: For a contradiction, let us first assume that

$$\mathrm{Q}_\tau\big[g(\theta, \cdot)|z\big] > g\big(\theta, \mathrm{Q}_\tau[w|z]\big) \equiv \hat{\alpha}.$$

This means that $\hat{\alpha} \notin \{\alpha \in \mathbb{R} : \Pr[\{w : g(\theta, w) \le \alpha\}|z] \ge \tau\}$, that is,

$$\Pr\big[\big\{w : g(\theta, w) \le \hat{\alpha}\big\}|z\big] < \tau.$$

Since $\hat{\alpha} = g(\theta, \mathrm{Q}_\tau[w|z])$ and g is non-decreasing in w, $\{w : w \le \mathrm{Q}_\tau[w|z]\} \subset \{w : g(\theta, w) \le \hat{\alpha}\}$. Thus, $\Pr[\{w : w \le \mathrm{Q}_\tau[w|z]\}|z] < \tau$, but this contradicts Lemma A.1(iv).

Conversely, assume that

$$\mathrm{Q}_\tau\big[g(\theta, \cdot)|z\big] < g\big(\theta, \mathrm{Q}_\tau[w|z]\big).$$

This means that there exists $\tilde{\alpha} < g(\theta, \mathrm{Q}_\tau[w|z])$ such that

$$\Pr\big[\big\{w : g(\theta, w) \le \tilde{\alpha}\big\}|z\big] \ge \tau.$$

Let $\tilde{w}$ be the supremum of the set $\{w : g(\theta, w) \le \tilde{\alpha}\}$. Since g is non-decreasing and left-continuous, $g(\theta, \tilde{w}) \le \tilde{\alpha}$. Moreover,

$$\Pr\big[\{w : w \le \tilde{w}\}|z\big] = \Pr\big[\big\{w : g(\theta, w) \le \tilde{\alpha}\big\}|z\big] \ge \tau.$$

Thus, $\tilde{w} \in \{\alpha \in \mathbb{R} : \Pr[\{w : w \le \alpha\}|z] \ge \tau\}$, which implies that $\tilde{w} \ge \mathrm{Q}_\tau[w|z]$. Thus, $\tilde{\alpha} \ge g(\theta, \tilde{w}) \ge g(\theta, \mathrm{Q}_\tau[w|z]) > \tilde{\alpha}$, which is a contradiction. *Q.E.D.*

The following corollary to the above lemma will be useful.

COROLLARY A.3: *Let* $T \in \mathbb{N} \cup \{\infty\}$, $h : \Theta \times \mathcal{Z}^T \times \mathcal{Z} \to \mathbb{R}$, $g : \Lambda \times \mathcal{Z}^T \times \mathcal{Z} \to \mathbb{R}$ *be non-decreasing and left-continuous in* $\mathcal{Z}$. *Then*,

$$\mathrm{Q}_\tau\big[h\big(\theta, z^T, \mathrm{Q}_\tau\big[g\big(\lambda, z^T, z_{t+1}\big)|z_t\big]\big)|z_1\big] = \mathrm{Q}_\tau\big[h\big(\theta, z^T, g\big(\lambda, z^T, \mathrm{Q}_\tau[z_{t+1}|z_t]\big)\big)|z_1\big].$$

PROOF: Let X denote the random variable $\mathrm{Q}_\tau[g(\lambda, z^t, z_{t+1})|z_t]$ and similarly, let Y denote $g(\lambda, z^t, \mathrm{Q}_\tau[z_{t+1}|z_t])$. Then, by Lemma A.2, $X = Y$. Therefore, $h(\theta, z^t, X) = h(\theta, z^t, Y)$ and the result follows. *Q.E.D.*

The following result will be useful below. It shows that an important property of comonotonic random variables is the behavior of the quantile functions of their sums.

PROPOSITION A.4: *Given the random variables* X *and* Y, *assume they are comonotonic*, *that is*, *that there exist random variable* Z *and continuous and increasing functions* h *and* g *such that* $X = h(Z)$ *and* $Y = g(Z)$. *Then* $\mathrm{Q}_\tau[X + Y] = \mathrm{Q}_\tau[X] + \mathrm{Q}_\tau[Y]$.

PROOF: Let Z, h, and g be as in the definition. Define $\tilde{h}(Z) \equiv h(Z) + g(Z)$. This function is clearly continuous and increasing. Therefore,

$$\begin{aligned} \mathrm{Q}_\tau[X + Y] &= \mathrm{Q}_\tau\big[\tilde{h}(Z)\big] = \tilde{h}\big(\mathrm{Q}_\tau[Z]\big) = h\big(\mathrm{Q}_\tau[Z]\big) + g\big(\mathrm{Q}_\tau[Z]\big) \\ &= \mathrm{Q}_\tau\big[h(Z)\big] + \mathrm{Q}_\tau\big[g(Z)\big] = \mathrm{Q}_\tau[X] + \mathrm{Q}_\tau[Y], \end{aligned}$$

by applying Lemma A.2 twice. *Q.E.D.*

A.2. *Proofs of Section 3*

PROOF OF THEOREM 3.4: This is essentially the same proof of Theorem 3.11, presented in detail below. Thus, we omit it. *Q.E.D.*

PROOF OF PROPOSITION 3.5: Let L be a bound for V^h. Using repeated times the recursive property (16), we can write

$$\begin{aligned} V^h(x, z) &= u\big(x_1^h, x_2^h, z_1\big) + \mathrm{Q}_\tau\big[\beta u\big(x_2^h, x_3^h, z_2\big) + \mathrm{Q}_\tau\big[\beta^2 u\big(x_3^h, x_4^h, z_3\big) + \cdots \\ &\quad + \mathrm{Q}_\tau\big[\beta^n u\big(x_{n+1}^h, x_{n+2}^h, z_n\big) + \beta^{n+1} V^h\big(x_n^h, Z_n\big)\big]|Z_n = z_n\big]\cdots|Z_1 = z\big] \\ &\leq u\big(x_1^h, x_2^h, z_1\big) + \mathrm{Q}_\tau\big[\beta u\big(x_2^h, x_3^h, z_2\big) + \mathrm{Q}_\tau\big[\beta^2 u\big(x_3^h, x_4^h, z_3\big) + \cdots \\ &\quad + \mathrm{Q}_\tau\big[\beta^n u\big(x_{n+1}^h, x_{n+2}^h, z_n\big) + \beta^{n+1} L\big]|Z_n = z_n\big]\cdots|Z_1 = z\big] \\ &= V^{h,n}(x, z) + \beta^{n+1} L, \end{aligned}$$

where in the last line we have used the property of quantiles that $\mathrm{Q}_\tau[X + \alpha] = \alpha + \mathrm{Q}_\tau[X]$ for $\alpha \in \mathbb{R}$; see Lemma A.2. Repeating the same argument with the lower bound $-L$, we can write

$$V^n(x, z) - \beta^{n+1} L \leq V^h(x, z) \leq V^{h,n}(x, z) + \beta^{n+1} L.$$

This concludes the proof. *Q.E.D.*

PROOF OF THEOREM 3.9: Assume that plans h and h' are such that $h_{t'}(\cdot) = h'_{t'}(\cdot)$ for all $t' \le t$ and $h' \succcurlyeq_{t+1,\Omega'_{t+1},x} h$ for all Ω'_{t+1}, x. From (12), this means that

$$V_{t+1}(h', x, z^{t+1}) \ge V_{t+1}(h, x, z^{t+1}), \quad \forall (x, z^t) \in \mathcal{X} \times \mathcal{Z}^{t+1}. \tag{49}$$

Therefore,

$$\begin{aligned} V_t(h', x, z^t) &= u(x_t^{h'}, x_{t+1}^{h'}, z_t) + \beta \mathrm{Q}_\tau[V_{t+1}(h', x, (Z^t, z_{t+1}))|Z^t = z^t] \\ &\ge u(x_t^{h'}, x_{t+1}^{h'}, z_t) + \beta \mathrm{Q}_\tau[V_{t+1}(h, x, (Z^t, z_{t+1}))|Z^t = z^t] \\ &= u(x_t^{h}, x_{t+1}^{h}, z_t) + \beta \mathrm{Q}_\tau[V_{t+1}(h, x, (Z^t, z_{t+1}))|Z^t = z^t] \\ &= V_t(h, x, z^t), \end{aligned}$$

where the first and last equalities come from the recursive equation (16), the first inequality comes from (49) and Lemma A.1(vi), while the equality in the third line comes from the fact that the plans agree on all times up to t, that is, $x_t^{h'} = x_t^h$ and $x_{t+1}^{h'} = h'_t(x_t^h, z^t) = h_t(x_t^h, z^t) = x_{t+1}^h$. This establishes the claim. *Q.E.D.*

PROOF OF THEOREM 3.11: We organize the proof in a series of lemmas.

LEMMA A.5: *If $v \in \mathcal{C}$, the map $(y, z) \mapsto \mathrm{Q}_\tau[v(y, w)|z]$ is continuous.*

PROOF: Consider a sequence $(y^n, z^n) \to (y^*, z^*)$. Since v and f are continuous, $v(y^n, w) \to v(y^*, w)$ and

$$m^n(\alpha) \equiv \Pr(\{w : v(y^n, w) \le \alpha\}|z^n) \to \Pr(\{w : v(y^*, w) \le \alpha\}|z^*) \equiv m^*(\alpha). \tag{50}$$

Let $\alpha^n \equiv \inf\{\alpha \in \mathbb{R} : m^n(\alpha) \ge \tau\} = \mathrm{Q}_\tau[v(y^n, \cdot)|z^n]$ and $\alpha^* \equiv \inf\{\alpha \in \mathbb{R} : m^*(\alpha) \ge \tau\} = \mathrm{Q}_\tau[v(y^*, \cdot)|z^*]$. We want to show that $\alpha^n \to \alpha^*$.

In general, $m^n(\cdot)$ and $m^*(\cdot)$ may fail to be continuous, but they are right-continuous and (weakly) increasing by Lemma A.1. Moreover, m^* and m^n are strictly increasing in the range of v. More precisely, for each y, define $R(y) \equiv \{\alpha \in \mathbb{R} : \exists w \text{ such that } v(y, w) = \alpha\}$. We claim that if $\alpha < \alpha'$, $\alpha, \alpha' \in R(y)$, then $m^*(\alpha') > m^*(\alpha)$, and similarly for m^n.[39]

Indeed, assume that $\exists w, w'$ such that $v(y, w) = \alpha$ and $v(y, w') = \alpha'$. The set $P = \{\alpha w + (1 - \alpha)w' : \alpha \in [0, 1]\}$ is contained in $\mathcal{Z}$ because this is convex. Thus, $\{v(y, p) : p \in P\}$ is connected, that is, a nonempty interval. We conclude that, since v is continuous, the set $\{w \in \mathcal{Z} : \alpha < v(y, w) < \alpha'\}$ is a nonempty and open set. (This implies, in particular, that $R(y)$ is an interval.) Since $f(\cdot|z)$ is strictly positive in $\mathcal{Z}$, we conclude that

$$m^*(\alpha') - m^*(\alpha) \ge \Pr(\{w \in \mathcal{Z} : \alpha < v(y, w) < \alpha'\}|z) > 0,$$

which establishes the claim. By Lemma A.1(iv), we have

$$m^n(\alpha^n) \ge \tau \quad \text{and} \quad m^*(\alpha^*) \ge \tau. \tag{51}$$

We will show that $\alpha^n \to \alpha^*$ by first establishing $\liminf_n \alpha^n \ge \alpha^*$ and then $\alpha^* \ge \limsup_n \alpha^n$.

[39]Note that m^n and m^* are the corresponding c.d.f. functions for v. Thus, proving that those functions are strictly increasing in the range of v leads to continuity of the quantile with respect to τ, by (an adaptation of) Lemma A.1(viii). But this is not what we need: we want continuity in (y, z). We prefer to offer here a direct and detailed argument, although long.

Suppose that $\underline{\alpha} \equiv \liminf_n \alpha^n < \alpha^*$. This means that there exist $\epsilon > 0$ and for each j, $n_j > j$ such that $\alpha^{n_j} < \underline{\alpha} + \epsilon < \alpha^*$. By the definition of α^*, $\underline{\alpha} + \epsilon < \alpha^*$ implies $m^*(\underline{\alpha} + \epsilon) < \tau$. However, by (51), $m^{n_j}(\alpha^{n_j}) \geq \tau$, which implies $m^{n_j}(\underline{\alpha} + \epsilon) \geq \tau$ and $m^*(\underline{\alpha} + \epsilon) \geq \tau$, by (50). This contradiction establishes that $\liminf_n \alpha^n \geq \alpha^*$.

If $\bar{\alpha} \equiv \limsup_n \alpha^n > \alpha^*$, there exist $\epsilon > 0$ and for each j, $n_j > j$ such that

$$\bar{\alpha} + \epsilon > \alpha^{n_j} > \bar{\alpha} - \epsilon > \bar{\alpha} - 2\epsilon > \alpha^* + \epsilon. \tag{52}$$

Recall that $\alpha^n = \inf\{\alpha \in \mathbb{R} : m^n(\alpha) \geq \tau\}$. Therefore, $\alpha^{n_j} > \bar{\alpha} - \epsilon$ implies $m^{n_j}(\bar{\alpha} - \epsilon) < \tau$. Thus, $m^{n_j}(\alpha^* + \epsilon) < m^{n_j}(\bar{\alpha} - \epsilon) < \tau$. This implies that

$$m^*(\alpha^*) \leq m^*(\bar{\alpha} - 2\epsilon) \leq m^*(\bar{\alpha} - \epsilon) = \lim_n m^{n_j}(\bar{\alpha} - \epsilon) \leq \tau \leq m^*(\alpha^*).$$

Therefore, m^* is constant between α^* and $\bar{\alpha} - 2\epsilon$. This will be a contradiction if we show that $\alpha^*, \bar{\alpha} - 2\epsilon \in R(y^*)$.

Since $m^*(\alpha^*) = \Pr(\{w : v(y^*, w) \leq \alpha^*\}|z^*) \geq \tau > 0$, $\{w : v(y^*, w) \leq \alpha^*\} \neq \emptyset$ and there exists some $\alpha \in R(y^*) \cap (-\infty, \alpha^*]$. On the other hand, if $\{w : \bar{\alpha} - 2\epsilon \leq v(y^*, w) \leq \bar{\alpha} + 2\epsilon\} = \emptyset$, then for sufficiently high j, $\{w : \bar{\alpha} - \epsilon \leq v(y^{n_j}, w) \leq \bar{\alpha} + \epsilon\} = \emptyset$. In this case, $m^{n_j}(\bar{\alpha} - \epsilon) = m^{n_j}(\bar{\alpha} + \epsilon) \equiv \tau^{n_j}$. But this would imply either $\alpha^{n_j} \leq \bar{\alpha} - \epsilon$, if $\tau^{n_j} \geq \tau$, or $\alpha^{n_j} \geq \bar{\alpha} + \epsilon$, if $\tau^{n_j} < \tau$. In either case, we have a contradiction with $\alpha^{n_j} \in (\bar{\alpha} - \epsilon, \bar{\alpha} + \epsilon)$ as required in (52). This contradiction shows that there exists $\alpha' \in R(y^*) \cap [\bar{\alpha} - 2\epsilon, \bar{\alpha} + 2\epsilon]$. Since $\alpha, \alpha' \in R(y^*)$, we have $[\alpha^*, \bar{\alpha} - 2\epsilon] \subset [\alpha, \alpha'] \subset R(y^*)$. This concludes the proof. *Q.E.D.*

LEMMA A.6: *For each $v \in \mathcal{C}$, the supremum in* (26) *is attained and $\mathbb{M}^\tau(v) \in \mathcal{C}$. Moreover, the optimal correspondence $Y : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{X}$ defined by*

$$Y(x, z) \equiv \arg \max_{y \in \Gamma(x,z)} \mathrm{Q}_\tau\big[u(x, y, z) + \beta v^\tau(y, w)|z\big] \tag{53}$$

is nonempty and upper semi-continuous.

PROOF: Let

$$g(x, y, z, w) = u(x, y, z) + \beta v(y, w). \tag{54}$$

By Lemma A.2, $\mathrm{Q}_\tau[g(x, y, z, \cdot)|z] = u(x, y, z) + \beta \mathrm{Q}_\tau[v(y, \cdot)|z]$. By Lemma A.5, $\mathrm{Q}_\tau[g(x, y, z, \cdot)|z]$ is continuous in (x, y, z). From Berge's Maximum Theorem, the maximum is attained, the value function $\mathbb{M}^\tau(v)$ is continuous, and Y is nonempty and upper semi-continuous. $\mathbb{M}^\tau(v)$ is bounded because u and v, hence g, are bounded. Therefore, $\mathbb{M}^\tau(v) \in \mathcal{C}$. *Q.E.D.*

We conclude the proof of Theorem 3.11 by showing that $\mathbb{M}^\tau$ satisfies Blackwell's sufficient conditions for a contraction.

LEMMA A.7: *$\mathbb{M}^\tau$ satisfies the following conditions*:
(a) *For any $v, v' \in \mathcal{C}$, $v \leq v'$ implies $\mathbb{M}^\tau(v) \leq \mathbb{M}^\tau(v')$.*
(b) *For any $a \geq 0$ and $x \in X$, $\mathbb{M}(v + a)(x) \leq \mathbb{M}(v)(x) + \beta a$, with $\beta \in (0, 1)$.*

Then, $\|\mathbb{M}(v)-\mathbb{M}(v')\| \le \beta\|v-v'\|$, *that is*, $\mathbb{M}$ *is a contraction with modulus* β. *Therefore*, $\mathbb{M}^\tau$ *has a unique fixed point* $v^\tau \in \mathcal{C}$.

PROOF: To see (a), let $v, v' \in \mathcal{C}$, $v \le v'$ and define g as in (54) and analogously for g', that is, $g'(x,y,z,w) = u(x,y,z) + \beta v'(y,w)$. It is clear that $g \le g'$. Then, by Lemma A.1(vi), $\mathrm{Q}_\tau[g(\cdot)|z] \le \mathrm{Q}_\tau[g'(\cdot)|z]$, which implies (a).

To verify (b), we use the monotonicity property (Lemma A.2):

$$\mathrm{Q}_\tau\big[u(x,y,z)+\beta\big(v(x,z)+a\big)|z\big] = \mathrm{Q}_\tau\big[u(x,y,z)+\beta v(x,z)|z\big]+\beta a.$$

Thus, $\mathbb{M}^\tau(v+a) = \mathbb{M}^\tau(v)+\beta a$, that is, (b) is satisfied with equality. *Q.E.D.*

PROOF OF THEOREM 3.12: We organize the proof in a series of lemmas. Let Assumption 2 hold. It is convenient to introduce the following notation. Let $\mathcal{C}' \subset \mathcal{C}$ be the set of the continuous functions $v : \mathcal{X}\times\mathcal{Z} \to \mathbb{R}$ which are concave in x and nondecreasing in z. It is easy to see that $\mathcal{C}'$ is a closed subset of $\mathcal{C}$. Let $\mathcal{C}'' \subset \mathcal{C}'$ be the subset of strictly concave functions. If we show that $\mathbb{M}^\tau(\mathcal{C}') \subset \mathcal{C}''$, then the fixed point of $\mathbb{M}^\tau$ will be strictly concave in x. (See, for instance, Stokey, Lucas, and Prescott (1989, Corollary 1, p. 52).)

LEMMA A.8: *Let Assumption* 2 *hold*. $\mathbb{M}^\tau(\mathcal{C}') \subseteq \mathcal{C}''$. *Therefore*, $v^\tau \in \mathcal{C}''$. *Moreover*, *the optimal correspondence* $Y : \mathcal{X}\times\mathcal{Z} \rightrightarrows \mathcal{X}$ *defined by* (53) *is single-valued*. *Therefore*, *we can denote it by a function* $y^*(x,z)$.

PROOF: Let $\alpha \in (0,1)$, $v \in \mathcal{C}'$ and consider $x_0, x_1 \in \mathcal{X}$, $x_0 \ne x_1$. For $i = 0,1$, let $y_i \in \Gamma(x_i, z)$ attain the maximum, that is,

$$\mathbb{M}^\tau(v)(x_i,z) = u(x_i,y_i,z)+\beta\mathrm{Q}_\tau\big[v(y_i,w)|z\big] = \mathrm{Q}_\tau\big[g(x_i,y_i,z,w)|z\big],$$

where g is defined by (54).

Let $x_\alpha \equiv \alpha x_0 + (1-\alpha)x_1$ and $y_\alpha \equiv \alpha y_0 + (1-\alpha)y_1$. First, let us observe that

$$\begin{aligned} g(x_\alpha,y_\alpha,z,w) &= u(x_\alpha,y_\alpha,z)+\beta v(y_\alpha,w) \\ &> \alpha u(x_0,y_0,z)+(1-\alpha)u(x_1,y_1,z) \\ &\quad + \beta v(y_\alpha,w) \\ &\ge \alpha u(x_0,y_0,z)+(1-\alpha)u(x_1,y_1,z) \\ &\quad + \beta\big[\alpha v(y_0,w)+(1-\alpha)v(y_1,w)\big] \\ &= \alpha g(x_0,y_0,z,w)+(1-\alpha)g(x_1,y_1,z,w), \end{aligned}$$

where the first inequality comes from the strict concavity of u and the second, from the concavity of v. That is, g is strictly quasiconcave, which establishes that $Y(x,z)$ is single-valued. Therefore,

$$\mathrm{Q}_\tau\big[g(x_\alpha,y_\alpha,z,w)|z\big] > \mathrm{Q}_\tau\big[\alpha g(x_0,y_0,z,w)+(1-\alpha)g(x_1,y_1,z,w)|z\big].$$

Note that the variables $X = g(x_0,y_0,z,w)$ and $Y = g(x_1,y_1,z,w)$ satisfy the assumption of Proposition A.4 since v is non-decreasing in w (holding z fixed). Therefore,

$$\begin{aligned} \mathrm{Q}_\tau\big[g(x_\alpha,y_\alpha,z,w)|z\big] &> \alpha\mathrm{Q}_\tau\big[g(x_0,y_0,z,w)|z\big]+(1-\alpha)\mathrm{Q}_\tau\big[g(x_1,y_1,z,w)|z\big] \\ &= \alpha\mathbb{M}^\tau(v)(x_0,z)+(1-\alpha)\mathbb{M}^\tau(v)(x_1,z). \end{aligned} \tag{55}$$

Therefore,

$$\mathbb{M}^\tau(v)(x_\alpha, z) \geq \mathrm{Q}_\tau\big[g(x_\alpha, y_\alpha, z, w)|z\big]$$
$$> \alpha\mathbb{M}^\tau(v)(x_0, z) + (1-\alpha)\mathbb{M}^\tau(v)(x_1, z).$$

This establishes strict concavity, concluding the proof. *Q.E.D.*

LEMMA A.9: *Let Assumption 2 hold. If $h : \mathcal{Z} \to \mathbb{R}$ is weakly increasing and $z \leq z'$, then* $\mathrm{Q}_\tau[h(w)|z] \leq \mathrm{Q}_\tau[h(w)|z']$.

PROOF: From Assumption 2(ii), if $h : \mathcal{Z} \to \mathbb{R}$ is weakly increasing and $z \leq z'$,

$$\int_{\mathcal{Z}} [-1_{\{\alpha\in\mathcal{Z}:h(\alpha)\leq w\}}] f(\alpha|z)\, d\alpha \leq \int_{\mathcal{Z}} [-1_{\{\alpha\in\mathcal{Z}:h(\alpha)\leq w\}}] f\big(\alpha|z'\big)\, d\alpha.$$

Thus,

$$\int_{\{\alpha\in\mathcal{Z}:h(\alpha)\leq w\}} f(\alpha|z)\, d\alpha \geq \int_{\{\alpha\in\mathcal{Z}:h(\alpha)\leq w\}} f\big(\alpha|z'\big)\, d\alpha. \tag{56}$$

If we define $H(w|z) = \Pr([h(W) \leq w]|Z = z)$, then (56) can be written as

$$H(w|z) \geq H\big(w|z'\big).$$

Observe that $\mathrm{Q}_\tau[h(w)|z] = \inf\{\alpha \in \mathbb{R} : H(\alpha|z) \geq \tau\}$ and, whenever $z \leq z'$, $H(w|z') \leq H(w|z)$, for all w. Therefore, if $z \leq z'$, then

$$\big\{\alpha \in \mathbb{R} : H(\alpha|z) \geq \tau\big\} \supset \big\{\alpha \in \mathbb{R} : H\big(\alpha|z'\big) \geq \tau\big\},$$

which implies that

$$\mathrm{Q}_\tau\big[h(w)|z\big] = \inf\big\{\alpha \in \mathbb{R} : H(\alpha|z) \geq \tau\big\} \leq \inf\big\{\alpha \in \mathbb{R} : H\big(\alpha|z'\big) \geq \tau\big\} = \mathrm{Q}_\tau\big[h(w)|z'\big],$$

as we wanted to show. *Q.E.D.*

LEMMA A.10: *Let Assumption 2 hold. If $v \in \mathcal{C}$ is increasing in z, then $\mathbb{M}^\tau(v)$ is strictly increasing in z.*

PROOF: Let $z_1, z_2 \in \mathcal{Z}$, with $z_1 < z_2$. For $i = 1, 2$, let $y_i \in \Gamma(x, z_i)$ realize the maximum, that is,

$$\mathbb{M}^\tau(v)(x_i, z) = u(x, y_i, z_i) + \beta\mathrm{Q}_\tau\big[v(y_i, w)|z_i\big].$$

Since u is strictly increasing in z, we have

$$\mathbb{M}^\tau(v)(x, z_1) = u(x, y_1, z_1) + \beta\mathrm{Q}_\tau\big[v(y_1, w)|z_1\big] < u(x, y_1, z_2) + \beta\mathrm{Q}_\tau\big[v(y_1, w)|z_1\big].$$

From Lemma A.9, we have $\mathrm{Q}_\tau[v(y_1, w)|z_1] \leq \mathrm{Q}_\tau[v(y_1, w)|z_2]$, which gives

$$\mathbb{M}^\tau(v)(x, z_1) < u(x, y_1, z_2) + \beta\mathrm{Q}_\tau\big[v(y_1, w)|z_2\big].$$

From Assumption 2, $\Gamma(x, z) \subseteq \Gamma(x, z')$, that is, $y_1 \in \Gamma(x, z_2)$. Optimality thus implies that

$$u(x, y_1, z_2) + \beta\mathrm{Q}_\tau\big[v(y_1, w)|z_2\big] \leq u(x, y_2, z_2) + \beta\mathrm{Q}_\tau\big[v(y_2, w)|z_2\big] = \mathbb{M}^\tau(v)(x, z_2).$$

Therefore, $\mathbb{M}^\tau(v)(x, z_1) < \mathbb{M}^\tau(v)(x, z_2)$, which shows strict increasingness in z. *Q.E.D.*

We conclude the proof of Theorem 3.12 by showing differentiability of v, which follows from an easy adaptation of Benveniste and Scheinkman's (1979) argument. For completeness and reader's convenience, we reproduce it here. Given (x, z), let $y^*(x, z) \in \Gamma(x, z)$ be unique maximum as established in Lemma A.8. Thus, for all (x, z), we have

$$v(x, z) = u\big(x, y^*(x, z), z\big) + \beta \mathrm{Q}_\tau\big[v\big(y^*(x, z), w\big)|z\big].$$

Fix x_0 in the interior of X and define

$$\bar{w}(x, z) = u\big(x, y^*(x_0, z), z\big) + \beta \mathrm{Q}_\tau\big[v\big(y^*(x_0, z), w\big)|z\big].$$

From the optimality, for a neighborhood of x_0, we have $\bar{w}(x, z) \leq v(x, z)$, with equality at $x = x_0$, which implies $\bar{w}(x, z) - \bar{w}(x_0, z) \leq v(x, z) - v(x_0, z)$. Note that $\bar{w}$ is concave and differentiable in x because u is. Thus, any subgradient p of v at x_0 must satisfy

$$p \cdot (x - x_0) \geq v(x, z) - v(x_0, z) \geq \bar{w}(x, z) - \bar{w}(x_0, z).$$

Thus, p is also a subgradient of $\bar{w}$. But since $\bar{w}$ is differentiable, p is unique. Therefore, v is a concave function with a unique subgradient. Therefore, it is differentiable in x (cf. Rockafellar (1970, Theorem 25.1, p. 242)) and its derivative with respect to x is the same as that of $\bar{w}$, that is,

$$\frac{\partial v^\tau}{\partial x_i}(x, z) = \frac{\partial \bar{w}}{\partial x_i}(x, z) = \frac{\partial u}{\partial x_i}\big(x, y^*(x, z), z\big),$$

as we wanted to show. *Q.E.D.*

PROOF OF LEMMA 3.14: By Stokey, Lucas, and Prescott (1989, Theorem 7.6), Γ has a measurable selection. Therefore, the argument in Stokey, Lucas, and Prescott (1989, Lemma 9.1) establishes the result. *Q.E.D.*

We need the following notation in the next proof. Let $T \in \mathbb{N} \cup \{\infty\}$ and $S : \mathcal{Z}^T \to \mathcal{Z}^{T-1}$ be the shift operator, that is, given $z = (z_1, z_2, \ldots, z_T) \in \mathcal{Z}^T$, $S(z) = (z_2, \ldots, z_T) \in \mathcal{Z}^{T-1}$. Abusing notation, let $S : H \to H$ also denote the shift operator for plans, that is, given $h \in H$, $h^s = S(h) \in H$ is defined as follows: for each given $z^\infty \in \mathcal{Z}^\infty$, $h^s_t(x, S(z^{t+1})) = h_{t+1}(x, z^{t+1})$. Let $S_t : H \to H$ be the composition of S with itself t times.

PROOF OF LEMMA 3.15: Let $t \geq 2$ (otherwise there is nothing to prove). Since $H_t(x, z) \subset H_1(x, z) = H(x, z)$ by definition, we have $v^*_t(x, z) \leq v_1(x, z)$. Suppose, for an absurd, that there exists $h \in H(x, z)$ such that

$$V_1(h, x, z) > v^*_t(x, z). \tag{57}$$

Let $\tilde{h}$ and $(\tilde{x}, \tilde{z}^t)$ be such that $S_{t-1}(\tilde{h}) = h$, $x^{\tilde{h}}_t(\tilde{x}, \tilde{z}^t) = x$ and $\tilde{z}_t = z$. Then, $V_t(\tilde{z}, \tilde{x}, \tilde{z}^t) = V_1(h, x, z)$. Since $v^*_t(x, z) \geq V_t(\tilde{z}, \tilde{x}, \tilde{z}^t)$, this establishes a contradiction with (57). *Q.E.D.*

PROOF OF LEMMA 3.16: If v is bounded and satisfies (28), then it is the unique fixed point of the contraction $\mathbb{M}^\tau$. Thus, the proof of Theorem 3.11 establishes, via the maximum theorem, the claims. *Q.E.D.*

PROOF OF PROPOSITION 3.17: Assume that v satisfies (28). It is sufficient to show that (i) $v(x,z) \geq V_1(h,x,z)$ for any $h \in H(x,z)$ and $(x,z) \in \mathcal{X} \times \mathcal{Z}$; and (ii) $v(x,z) = V_1(h^\psi, x, z)$. Let $h \in H(x,z)$. We have

$$\begin{aligned} v(x,z) &= \sup_{y \in \Gamma(x_1^h, z_1)} u(x_1^h, y, z_1) + \beta \mathrm{Q}_\tau[v(y, z_2)|z_1] \\ &\geq u(x_1^h, x_2^h, z_1) + \beta \mathrm{Q}_\tau[v(x_2^h, z_2)|z_1] \\ &= u(x_1^h, x_2^h, z_1) + \beta \mathrm{Q}_\tau\Big[\sup_{y \in \Gamma(x_2^h, z_2)} \{u(x_2^h, y, z_2) + \beta \mathrm{Q}_\tau[v(y, z_3)|z_2]\}\Big|z_1\Big] \\ &\geq u(x_1^h, x_2^h, z_1) + \mathrm{Q}_\tau[\beta u(x_2^h, x_3^h, z_2) + \mathrm{Q}_\tau[\beta^2 v(x_3^h, z_3)|z_2]|z_1], \end{aligned}$$

where the two inequalities come from the definition of sup, and the equalities from (28) and Corollary A.3. Repeating the same arguments, we obtain

$$\begin{aligned} v(x,z) \geq{}& u(x_1^h, x_2^h, z_1) + \mathrm{Q}_\tau[\beta u(x_2^h, x_3^h, z_2) + \mathrm{Q}_\tau[\beta^2 u(x_3^h, x_4^h, z_3) + \cdots \\ &+ \mathrm{Q}_\tau[\beta^n u(x_{n+1}^h, x_{n+2}^h, z_n) + \beta^{n+1} v(x_n^h, Z_n)]|Z_n = z_n]\cdots|Z_1 = z]. \end{aligned}$$

Repeating the arguments in the proof of Proposition 3.5, we can conclude that the limit of the right-hand side when $n \to \infty$ is $V^h(x,z) = V_1(h,x,z)$. Thus, we have established that $v(x,z) \geq V_1(h,x,z)$. Since h was arbitrary, then $v(x,z) \geq v^*(x,z)$. On the other hand, for h^ψ the inequalities above hold with equality and we obtain $v(x,z) = v^*(x,z)$. *Q.E.D.*

PROOF OF THEOREM 3.18: Let $g(x,y,z,w) \equiv u(x,y,z) + \beta \mathrm{Q}_\tau[v^\tau(y,w)|z]$ and $y^*(x,z)$ be an interior solution of the problem (28). Observe that v^τ is increasing in w, differentiable in its first variable, and for $0 < x_i' - x_i < \epsilon$, for some small $\epsilon > 0$,

$$v^\tau(x_i', x_{-i}, z) - v^\tau(x_i, x_{-i}, z) = \int_x^{x'} \frac{\partial v^\tau}{\partial x_i}(\alpha, x_{-i}, z)\, d\alpha = \int_x^{x'} \frac{\partial u}{\partial x_i}(\alpha, x_{-i}, z)\, d\alpha$$

is increasing in z because $\frac{\partial u}{\partial x_i}$ is. Therefore, the assumptions of Proposition 3.19 are satisfied and we conclude that $\frac{\partial \mathrm{Q}_\tau}{\partial x_i}[v^\tau(x,z)] = \mathrm{Q}_\tau[\frac{\partial v^\tau}{\partial x_i}(x,z)]$. Since u is differentiable in y, so is g. Since $y^*(x,z)$ is interior, the following first-order condition holds:

$$\frac{\partial g}{\partial y_i}(x, y^*(x,z), z, \mathrm{Q}_\tau[w|z]) = \frac{\partial u}{\partial y_i}(x, y^*(x,z), z) + \beta \mathrm{Q}_\tau\left[\frac{\partial v^\tau}{\partial x_i}(y^*(x,z), w)\Big| z\right] = 0.$$

Now we apply Theorem 3.12 and its expression: $\frac{\partial v^\tau}{\partial x_i}(x,z) = \frac{\partial u}{\partial x_i}(x, y^*(x,z), z)$, to conclude that

$$\frac{\partial u}{\partial y_i}(x, y^*(x,z), z) + \beta \mathrm{Q}_\tau\left[\frac{\partial u}{\partial x_i}(y^*(x,z), y^*(y^*(x,z), w), w)\Big| z\right] = 0. \tag{58}$$

Now, we have just to put the notation of a sequence. For this, let $h = (x_t)$ denote an optimal path beginning at (x_0, z_0); (58) can be rewritten, substituting x for x_t^h, $y^*(x,z)$ for x_{t+1}^h, $y^*(y^*(x,z), w)$ for x_{t+2}^h, z for z_t, and w for z_{t+1}, as

$$\frac{\partial u}{\partial y_i}(x_t^h, x_{t+1}^h, z_t) + \beta \mathrm{Q}_\tau\left[\frac{\partial u}{\partial x_i}(x_{t+1}^h, x_{t+2}^h, z_{t+1})\Big| z_t\right] = 0, \tag{59}$$

which we wanted to establish. *Q.E.D.*

PROOF OF PROPOSITION 3.19: Fix a sufficiently small $\delta > 0$ and $x = (x_i, x_{-i})$, with the usual meaning.[40] Define $X = d(z) \equiv h(x_i + \delta, x_{-i}, z) - h(x_i, x_{-i}, z)$ and $Y = \tilde{g}(z) = h(x_i, x_{-i}, z)$. Since h and $d(z) \equiv h(x_i + \delta, x_{-i}, z) - h(x_i, x_{-i}, z)$ are increasing in z by assumption, the random variables X and Y satisfy the assumptions of Proposition A.4, which allows us to conclude that

$$\begin{aligned} \mathrm{Q}_\tau\big[h(x_i + \delta, x_{-i}, z)\big] &= \mathrm{Q}_\tau[X + Y] = \mathrm{Q}_\tau[X] + \mathrm{Q}_\tau[Y] \\ &= \mathrm{Q}_\tau\big[h(x_i + \delta, x_{-i}, z) - h(x_i, x_{-i}, z)\big] + \mathrm{Q}_\tau\big[h(x_i, x_{-i}, z)\big]. \end{aligned}$$

Therefore, for all sufficiently small $\delta > 0$,

$$\frac{\mathrm{Q}_\tau\big[h(x_i + \delta, x_{-i}, z)\big] - \mathrm{Q}_\tau\big[h(x_i, x_{-i}, z)\big]}{\delta} = \mathrm{Q}_\tau\left[\frac{h(x_i + \delta, x_{-i}, z) - h(x_i, x_{-i}, z)}{\delta}\right].$$

Since $\delta \mapsto \frac{h(x_i+\delta, x_{-i}, z) - h(x_i, x_{-i}, z)}{\delta}$ is continuous, Lemma A.5 implies that

$$\begin{aligned} \lim_{\delta \downarrow 0} \frac{\mathrm{Q}_\tau\big[h(x_i + \delta, x_{-i}, z)\big] - \mathrm{Q}_\tau\big[h(x_i, x_{-i}, z)\big]}{\delta} &= \lim_{\delta \downarrow 0} \mathrm{Q}_\tau\left[\frac{h(x_i + \delta, x_{-i}, z) - h(x_i, x_{-i}, z)}{\delta}\right] \\ &= \mathrm{Q}_\tau\left[\lim_{\delta \downarrow 0} \frac{h(x_i + \delta, x_{-i}, z) - h(x_i, x_{-i}, z)}{\delta}\right] \\ &= \mathrm{Q}_\tau\left[\frac{\partial h}{\partial x_i}(x, z)\right]. \end{aligned}$$

We can adapt the above arguments for $\delta > 0$ and $X = d(z) = h(x_i, x_{-i}, z) - h(x_i - \delta, x_{-i}, z)$ and $Y = \tilde{g}(z) = h(x_i, x_{-i}, z)$ to conclude that

$$\begin{aligned} \lim_{\delta \downarrow 0} \frac{\mathrm{Q}_\tau\big[h(x_i, x_{-i}, z)\big] - \mathrm{Q}_\tau\big[h(x_i - \delta, x_{-i}, z)\big]}{\delta} &= \lim_{\delta \downarrow 0} \mathrm{Q}_\tau\left[\frac{h(x_i, x_{-i}, z) - h(x_i - \delta, x_{-i}, z)}{\delta}\right] \\ &= \mathrm{Q}_\tau\left[\lim_{\delta \downarrow 0} \frac{h(x_i, x_{-i}, z) - h(x_i - \delta, x_{-i}, z)}{\delta}\right] \\ &= \mathrm{Q}_\tau\left[\frac{\partial h}{\partial x_i}(x, z)\right]. \end{aligned}$$

By changing $\delta > 0$ above by $\tilde{\delta} = -\delta < 0$, we obtain

$$\lim_{\tilde{\delta} \uparrow 0} \frac{\mathrm{Q}_\tau\big[h(x_i + \tilde{\delta}, x_{-i}, z)\big] - \mathrm{Q}_\tau\big[h(x_i, x_{-i}, z)\big]}{\tilde{\delta}} = \mathrm{Q}_\tau\left[\frac{\partial h}{\partial x_i}(x, z)\right].$$

This shows that the right and left derivative of $x_i \mapsto \mathrm{Q}_\tau[h(x_i, x_{-i}, z)]$ exist and are equal. Therefore, $x_i \mapsto \mathrm{Q}_\tau[h(x_i, x_{-i}, z)]$ is differentiable and its derivative is $\mathrm{Q}_\tau[\frac{\partial h}{\partial x_i}(x, z)]$, as we wanted to show. *Q.E.D.*

[40]"Sufficiently small" here means that $\delta > 0$ is taken so that $d(z) \equiv h(x_i + \delta, x_{-i}, z) - h(x_i, x_{-i}, z)$ is increasing, as required by the assumption of the proposition. This "smallness" condition will be left implicit below.

A.3. *Proofs of Section 4*

PROOF OF LEMMA 4.1: Assumption 1(i)–(iii) and (v) are immediate. Since $\mathcal{Z}$ and $\mathcal{X}$ are bounded, and U and $z \mapsto z + p(z)$ are C^1, u is C^1 and bounded. Thus, Assumption 1 is satisfied. Similarly, Assumptions 2 are easily seen to be satisfied. It remains to verify the assumption of Theorem 3.18, namely, that $\frac{\partial u}{\partial x_i}(x_t^h, x_{t+1}^h, z_t)$ is strictly increasing in z_t, which happens if and only if $\log \frac{\partial u}{\partial x_i}(x_t^h, x_{t+1}^h, z_t)$ is strictly increasing in z_t. Since

$$\log \frac{\partial u}{\partial x}(x, y, z) = -\gamma \log\big[z \cdot x + p(z) \cdot (x - y)\big] + \log\big(z + p(z)\big),$$

and $x_t^h = x_{t+1}^h = 1$, we need to verify only that $-\gamma[\log(z)]' + [\log(z + p(z))]' > 0$. This is equivalent to $\gamma < z[\log(z + p(z))]'$, which is contained in Assumption 3(iv). *Q.E.D.*

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