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Characterization of bidding behavior in multi-unit auctions☆

Luciano I. de Castro[a], Alvaro Riascos[b,*]

[a] University of Illinois at Urbana-Champaign, 185 Wohlers Hall, 1206 S 6th Street, Champaign, IL 61820, USA
[b] Universidad de los Andes, Facultad de Economía and CEDE, Carrera 1 No. 18A-70, Bloque C, Bogota D.C., Colombia

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ABSTRACT

This paper considers a very general class of single or multi-unit auctions of indivisible objects. The model allows for interdependent values, multidiminensional types and any attitude towards risk. Assuming only optimal behavior, we prove that each bid is chosen in order to equalize the marginal benefit to the marginal cost of bidding. This generalizes many existing results in the literature. We use this characterization to obtain sufficient conditions for truthful bidding, monotonic best reply strategies and identification results for multi-unit auctions.

1. Introduction

Most results in auction theory rely on the assumption that bidders play equilibrium strategies. This assumption is fairly reasonable in most cases, but some empirical and experimental findings suggest that it is not always satisfied.[1] On the other hand, it seems that very little can be said out of equilibrium behavior. In this paper we show that a very weak assumption about bidders' behavior – that each bidder plays a best reply to the strategies that he believes the others are playing – can still provide useful conclusions.[2]

Our model encompasses a very general class of sealed-bid, single and multi-units auctions. We allow for interdependent values, asymmetric valuations, any attitude towards risk, non-monotonic value functions, non-separable transfers, dependent signals of any dimension, unitary or multiple unit demands auctions with just sellers, buyers or both. Also, no common prior is required. Under these general conditions we prove what we call the "basic principle of bidding". That is "a rational bidder bids in order to equalize the marginal benefit of bidding (the utility that he obtains in case of winning) to the marginal cost of bidding".

☆ This is a extended and improved version of the second chapter of de Castro (2004).

* Corresponding author. Tel.: +57 2 3394949; fax: +57 2 3324492.
E-mail addresses: decastro.luciano@gmail.com (L.I. de Castro), ariascos@uniandes.edu.co (A. Riascos).

1 See Kagel (1995) for findings in the experimental literature, and Laffont (1997) for a review of empirical works.

2 Thus, our behavioral assumption is even weaker than that of rationalizable strategies, as studied by Battigalli and Siniscalchi (2003).

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In a sense, it is obvious that in standard smooth optimization problems, at the optimum the marginal benefit (derivative of the objective function) equals to the marginal cost (shadow price) of the constraints. Nevertheless, this is not exactly the case for auctions, where the marginal cost does not come from a constraint. This is also different from the classic firm's problem, where revenue and costs are separable and the first order condition naturally gives that marginal revenue should equal the marginal cost. In contrast, the basic principle of bidding holds without assuming separability of revenues and costs. In auctions, the basic trade-off that a bidder faces is that a higher bid, although it increases the probability of winning, it may also decrease the payoff in case of winning. Using Leibniz rule to differentiate an integral that depends on the variable in the region of integration and in the integrand, we obtain two terms. These two terms can be interpreted as marginal benefit and marginal cost.

Although the proof is reminiscent of Leibniz rule in differential calculus, this result is not elementary because we need the theory of differentiation of measures. When we introduce additional assumptions, i.e., continuously differentiability of payoffs at the optimum bid, we provide first order conditions that generalize those obtained by Milgrom and Weber (1982) for first- and second-price auctions, Krishna and Morgan (1997) for the all-pay auction and war of attrition, and Williams (1991) for buyers'-bids double auctions. Under risk neutrality, symmetry and monotonicity of the utility function, the provided characterization reduces to the ones in those papers. In addition, in a indivisible good framework, with or without interdependent values, we provide first order conditions for the multi-unit discriminatory, uniform and Vickrey auction. To the extent of our knowledge, the indivisible goods, interdependent values model caracterization for the discriminatory auction is new.

The payoff characterization lemma, which is the main result of this paper, and which is valid in the most general setting, opens the way to a general approach to equilibrium existence for general auction models like in Araujo and de Castro (2008). It can also provide insights for empirical and experimental studies, since every bid (even the initial or the apparently inconsistent ones in a repeated game) bears valuable information about players' beliefs. The first order conditions is a first step towards characterizing rational behavior in general auctions. Also, as the recent literature on econometric identification of auction models has pointed out, characterizing best reply bidding strategies allows for identification in many standard auction formats.[3] Following this approach, our result allows for general econometric identification of multidimensional auction models. We present some of these results in Section 4.3. We also use our results to give sufficient conditions for truthful bidding in multi-unit auctions. This gives an immediate proof that Vickrey auction is truth-telling, and that only the bid for the first unit in uniform auctions is truthful.

The paper is organized as follows. Section 2 presents the model and notation. Section 3 contains the main results and some examples of direct applications. Some applications are provided in Section 4. First, we give sufficient conditions for truthful bidding and provide examples for when it holds and when it does not. We also prove a monotonic-best reply result that generalizes for multiunits auctions a result of Araujo and de Castro (2008), which is the key result for their proof of equilibrium existence in single-object auctions. Yet in Section 4 we report results on indentification in multiunit auctions, some of which are new, to the best of our knowledge. Section 5 is a brief conclusion. Appendix A contains some of the more technical proofs.

2. The model

Our model is inspired in auction games, although it can encompass a general class of discontinuous games. For convenience, we will use the terminology of auction theory such as "bidders", "bidding functions" and "bids", for players, strategies and actions, respectively.

2.1. Players and information

There are N strategic bidders. We allow for the existence of an uninformed and non-strategic player, named 0. This is the seller in traditional auctions. For double auctions, there is no such player. We denote by $\mathcal{N} = \{0, 1, \ldots, N\}$ the set of all players (strategic and non-strategic). Player $i \in \mathcal{N}$ receives a signal (i.e., private information), $t_i \in T_i$ where T_i denotes player i's information set. We denote by $t = (t_1, t_2, \ldots, t_N) = (t_i, t_{-i})$ the vector of all players' information, where $t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_N)$, as usual. Let $\Im_i$ be a σ-field of subsets of T_i and define $T \equiv \prod_{i \in \mathcal{N}} T_i$ and the product σ-field over T, $\Im = \prod_{i \in \mathcal{N}} \Im_i$. Players beliefs are functions $\tau_i : T_i \to \Delta(T_{-i}, \Im_{-i})$, where $T_{-i} \equiv \prod_{j \neq i} T_j$, $\Im_{-i} = \prod_{i \neq j} \Im_i$ and $\Delta(T_{-i}, \Im_{-i})$ is the set of probability distribution over the measurable space $(T_{-i}, \Im_{-i})$. If g is a function of t_{-i}, we denote the expectation of g with respect to $\tau_i(t_i)$, by $E[g|t_i]$. Also to ease notation we will write $\tau(\cdot|t_i)$ instead of $\tau_i(t_i)$.

Notice that we do not assume that players beliefs need to represent conditional beliefs derived from a common prior over T. Individual signals may be dependent and of arbitrary dimension.

2.2. Objects and bidding

There are K indivisible objects. Each player $i \in \mathcal{N}$ comes to the auction with $e_i \in \{0, 1, 2, \ldots\}$ units of the same object, and $\sum_{i=0}^{N} e_i = K$. After receiving its signal, a strategic player submits a sealed proposal, that is, a bid (or offer) that is a vector of

[3] See Athey and Haile (2007) for a survey on the main issues regarding the econometric identification of auction models.

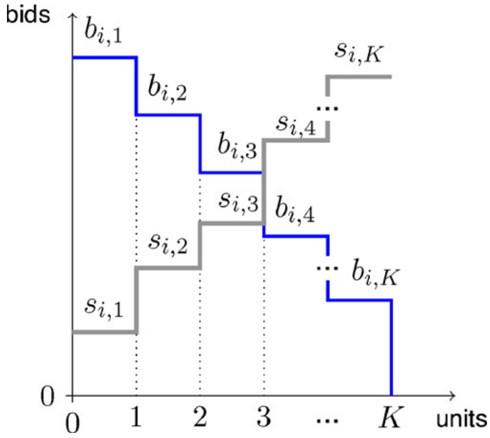

Fig. 1. Bid (b_i) and supply (s_i) curves for bidder i. In the situation displayed, bidder i receives three units, because $b_{i,3} > s_{i,3}$ but $b_{i,4} < s_{i,4}$.

real numbers, $b_i \in B \subseteq \mathbb{R}^K$ where B denotes the set of valid bids, that is, $B = \{b \in \mathbb{R}^K : b_k \geq b_{k+1} \text{ for } k = 1, \ldots, K-1\} \cap [\underline{b}, \bar{b}]$, $b_{i,k}$ is the maximum value that bidder i is willing to pay for the k th unit, given that he is receiving $k-1$ units; and $[\underline{b}, \bar{b}]$ denotes a K dimensional rectangle that bounds the set of all bids. Since bids are non-increasing we are implicitly assuming that there are no complementarity among objects. Bids are in units of account (i.e., dollars). The non-strategic player 0 also places a bid $b_0 \in B$, meaning that there is a reserve price for each unit.[4] For instance, in a one-object auction ($K = 1$) where all players are buyers, if $\max_{j=1,\ldots,N} b_{j,1} < b_{0,1}$, this means that none of the bidders are willing to pay the reserve price. The difference is that b_0 is known to everyone at the time the auction takes place, while b_j, $j \neq 0$, is not known to bidder $i \neq j$, $i \in \mathcal{N}$.[5] We denote by b the vector of all players' bids, $b \in \mathbb{R}^{(N+1)K}$.

2.3. Allocation and payoffs

The "auction house" computes the bids and determines how many units each player receives. If player i wins a k th unit, his payoff is increased by $u_{i,k}(t, b)$, where $u_{i,k} : T \times \mathbb{R}^{(N+1)K} \to \mathbb{R}$.[6] Thus, if player $i \in \mathcal{N}$ ends the auction with exactly $m_i \in \{0, 1, \ldots, K\}$ units, his payoff is $\sum_{k=0}^{m_i} u_{i,k}(t, b)$. The utility functions $u_{i,k}(t, b)$ incorporate in their definition the number of units e_i that bidder i has, as we explain in the examples below. If $k \leq e_i$, $u_{i,k}(t, b)$ stands for the utility of keeping the k th object and if $k > e_i$, $u_{i,k}(t, b)$ stands for the utility of buying the k th object. We will explain below the allocation rule, which will complete the definition of the payoff for each bid profile. In the examples we shall restrict to separable transfers so, for later reference, for each player i and unit k, let $v_{i,k} : T \to \mathbb{R}$ be a function such that $v_{i,k}(t)$ represents the (marginal) value, in units of account, of the k th unit for player i when the vector of signals is $t \in T$.

If $m_i < e_i$, the player has sold $e_i - m_i$ units in the auction and if $m_i > e_i$, the player has bought $m_i - e_i$ units in the auction. No negotiation was made if $m_i = e_i$.

Given b_{-i}, let $s_i = (s_{i,1}, s_{i,2}, \ldots, s_{i,K})$, with $s_{i,1} \leq s_{i,2} \leq \cdots \leq s_{i,K}$, denote the (inverse) residual supply curve facing bidder i. That is, $s_{i,K}$ is the highest of the bids by players $j \neq i$, $s_{i,K-1}$ is the second highest and so on. Thus, for getting (for sure) at least one unit, bidder i's highest bid must be above $s_{i,1}$, that is, $b_{i,1} > s_{i,1}$. For bidder i winning at least two units, it is necessary $b_{i,2} > s_{i,2}$ and so on. Fig. 1 illustrates this.

In order to decide who wins an object, we will assume that the auction house uses an allocation (or tie-breaking) rule.

Definition 1. An allocation rule is any function $a : \mathbb{R}^{(N+1)K} \to [0, 1]^{(N+1)K}$ such that:

1. If $b_{i,k} < s_{i,k}$ then $a_{i,k}(b) = 0$.
2. If $b_{i,k} > s_{i,k}$ then $a_{i,k}(b) = 1$.
3. The allocation rule is non-increasing in k, that is, for all $k' \leq k$, $a_{i,k'}(b) \geq a_{i,k}(b)$.[7]
4. $\sum_{i=0}^{N} \sum_{k=1}^{K} a_{i,k}(b) = K$.

[4] If the model does not specify a reserve price it is usual to assume $b_0 = 0$.

[5] Unknown reserve prices can be modeled as the bid of a strategic bidder.

[6] We consider the dependence on b instead of b_i because we want to include in our results auctions where the payoff depends on bids of the opponents, such as the second-price auction, for instance. Also, this allows the study of "exotic" auctions, i.e., auctions where the payment is an arbitrary function of all bids.

[7] If there is no tie, this condition follows from 1 and 2.

The interpretation is the following. If $a_{i,k}(b)=1$ then player i wins at least k objects. If $a_{i,k}(b)=0$ then player i wins at most $k-1$ objects. Formally, the first condition says that if player i's k th bid is lower than the $K-k+1$ highest competing bid he will not be awarded the k th object. The second condition says that if player i bids higher for unit k than the $K-k+1$ highest competing bids then he will win at least k objects. The third says that if he wins at least k objects then he must also win at least $1,\ldots,k-1$ objects. The fourth says that at most K units are allocated among the N agents.

Observe that in the definition of allocation rules, there is freedom to define the rule only when $b_{i,k}=s_{i,k}$, provided the other conditions are satisfied. Thus, it is sufficient to define the rule for ties.

This setting is very general and applies to a broad class of discontinuous games, as we exemplify below.[8]

2.3.1. Allocation rules

Example 1 *(Nominal allocation rule).* Let us suppose that the bidders are numbered following a given order (say, the lexicographic order for their names). We can define that, in the case of a tie, the bidder with the least number, among those that are tying, gets the object. It is easy to see that this rule satisfies all conditions in Definition 1.

Another example of allocation rule is the standard one, that splits randomly the objects.

Example 2 *(Standard allocation rule).* In the case of a tie, the objects involved in the tie are randomly divided among the tying bidders. Formally: if $b_{i,k}=s_{i,k}$ then $a_{i,k}(b)=p/q$ where p is the number of the objects to be allocated in the tie, that is, $p=K-\sharp\{(j,\tilde{k}):\text{such that}\, b_{j,\tilde{k}}>b_{i,k}\}$ and q is the number of tying bids, that is, $q=\sharp\{(j,\tilde{k}):\text{such that}\, b_{j,\tilde{k}}=b_{i,k}\}$.

2.3.2. Auctions

Example 3 *(Single-unit auctions).* $u_{i,1}(t,b)=U_i(v_{i,1}(t)-b_{i,1})$ and $u_{i,0}(t,b)=0$ corresponds to a first-price auction with risk aversion or risk loving.[9] If $U_i(x)=x$, we have risk neutrality. If $u_{i,1}(t,b)=v_{i,1}(t)$ and $u_{i,0}(t,b)=-b_{i,1}$ we have the all-pay auction. If $u_{i,1}(t,b)=v_{i,1}(t)-\mathbf{s}_{i,1}$ and $u_{i,0}(t,b)=0$ we have the second-price auction. If $u_{i,1}(t,b)=v_{i,1}(t)-\mathbf{s}_{i,2}$ and $u_{i,0}(t,b)=0$ we have the third-price auction. If $u_{i,1}(t,b)=v_{i,1}(t)+b_{i,1}-\mathbf{s}_{i,1}$ and $u_{i,0}(t,b)=-b_{i,1}$ we have the war of attrition. We can have also combinations of these games. For example, $u_{i,1}(t,b)=v_{i,1}(t)-\alpha b_{i,1}-(1-\alpha)\mathbf{s}_{i,1}$ and $u_{i,0}(t,b)=0$, with $\alpha\in(0,1)$, gives a combination of the first- and second-price auctions.

Example 4 *(Multi-unit auction with unitary demand).* It is also useful to consider K-unit auctions with unitary demand, among N buyers, $1<K<N$. In this case, $b_{j,k}<b_{0,1}$, for all $j=1,\ldots,N$ and $k=2,\ldots,K$. Then, a pay-your-bid auction is given by $u_{i,1}(t,b)=v_{i,1}(t)-b_{i,1}$ and $u_{i,0}(t,b)=0$. If it is a uniform price with the price determined by the highest looser's bid, $u_{i,1}(t,b)=v_{i,1}(t)-\mathbf{s}_{i,K}$ and $u_{i,0}(t,b)=0$. If it is a uniform price with the price determined by the lowest winner's bid, $u_{i,0}(t,b)=0$, $u_i(t,b)=v_{i,1}(t)-\max\{b_{i,1},\mathbf{s}_{i,K}\}$.

Example 5 *(Multi-unit auctions with multi-unit demand).* $u_{i,1}(t,b)=v_{i,1}(t)-b_{i,1},\ldots,u_{i,K}(t,b)=v_{i,K}(t)-b_{i,K}$ and $u_{i,0}(t,b)=0$ corresponds to a multiple unit auction with discriminatory price. If $u_{i,1}(t,b)=v_{i,1}(t)-p(b),\ldots,u_{i,K}(t,b)=v_{i,K}(t)-p(b)$ and $u_{i,0}(t,b)=0$ it correspond to a uniform multiple unit auction. There are two different uniform price auctions: $p(b)$ can be the lowest winner's bid (as in some actual treasury bills auctions) or $p(b)$ can be the highest looser's bid (as described by Krishna, 2002). If $u_{i,1}(t,b)=v_{i,1}(t)-\mathbf{s}_{i,1},\ldots,u_{i,k}(t,b)=v_{i,k}(t)-\mathbf{s}_{i,k}$, $u_{i,K}(t,b)=v_{i,K}(t)-\mathbf{s}_{i,K}$ and $u_{i,0}(t,b)=0$ we have Vickrey auction.

2.4. Strategies and order statistics

The strategy of a bidder $i\in\mathcal{N}$ is a bidding function $\mathbf{b}_i:T_i\to B$. We will use bold type for bidding functions. Notice that we do not specify a strategy for the non-strategic player, $i=0$. We will restrict to integrable strategies, that is, we assume that the vector of strategies is $\mathbf{b}=(\mathbf{b}_i)_{i\in\mathcal{N}}\in\prod_{i\in\mathcal{N}}\mathbb{L}^1(T_i,B)$. For a vector of strategies $\mathbf{b}=(\mathbf{b}_i)_{i\in\mathcal{N}}$, let $\mathbf{b}_{-i}$ be the vector of strategies of all strategic players except player i, we denote by $\mathbf{s}_i$, for a fixed $\mathbf{b}_{-i}$, the function $\mathbf{s}_i:T_{-i}\to\mathbb{R}^{KN}$ that orders the NK vector $\mathbf{b}_{-i}(t_{-i})$ from the highest to the lowest bid. Given $\mathbf{b}_{-i}$ and j, $1\le j\le KN$, define the distribution function, $F_{\mathbf{s}_{i,j}}(\cdot|t_i)$ on $\mathbb{R}$, by $F_{\mathbf{s}_{i,j}}(\beta|t_i)\equiv\tau(\{t_{-i}\in T_{-i}:\mathbf{s}_{i,j}(t_{-i})<\beta\}|t_i)$ and let $f_{\mathbf{s}_{i,j}}(b|t_i)$ be its Radon–Nykodim derivative with respect to the Lebesgue measure. We denote by $F^{\perp}_{\mathbf{s}_{i,j}}(b|t_i)$ the singular part of $F_{\mathbf{s}_{i,j}}(b|t_i)$.

In the examples, sometimes we will restrict to monotone strategies. In such cases we will implicitly assume that $T_i=\mathbb{R}$ and use the following notation. Given $t\in T$, we define $t_{(-i)}$ as $t_{(-i)}\equiv\max_{j\neq i}t_j$.

2.5. Expected payoff

In order to simplify notation below, we will write $(\cdot)$ in the place of $(t_i,t_{-i},b_i,\mathbf{b}_{-i}(t_{-i}))$; $(\beta,\cdot)$ in the place of $(t_i,t_{-i},(\beta,b_{i,-j}),\mathbf{b}_{-i}(t_{-i}))$; and $(\circ)$ in the place of $(b_i,\mathbf{b}_{-i}(t_{-i}))$. Thus, if the bid b_0 and the profile of bidding functions $\mathbf{b}_{-i}$

[8] Obviously, the utility functions are specified only for strategic bidders, that is, for $i\neq 0$.
[9] If we put $u_{i,1}(t,b)=U_i(v_{i,1}(t)-b_{i,1})$ we can have any attitude towards risk.

are fixed, the expected payoff of bidder i of type t_i, when bidding b_i, is:

$$\Pi_i(t_i, b_i, \mathbf{b}_{-i}) \equiv \int_{T_{-i}} u_{i,0}(\cdot)\tau(\mathrm{d}t_{-i}|t_i) + \sum_{k=1}^{K}\int_{T_{-i}} a_{i,k}(\circ)u_{i,k}(\cdot)\tau(\mathrm{d}t_{-i}|t_i),$$

which is equivalent to

$$\Pi_i(t_i, b_i, \mathbf{b}_{-i}) = \int_{T_{-i}} u_{i,0}(\cdot)\tau(\mathrm{d}t_{-i}|t_i) + \sum_{k=1}^{K}\int_{T_{-i}} u_{i,k}(\cdot)1_{[b_{i,k}>\mathbf{s}_{i,k}]}\tau(\mathrm{d}t_{-i}|t_i) + \sum_{k=1}^{K}\int_{T_{-i}} a_{i,k}(\circ)u_{i,k}(\cdot)1_{[b_{i,k}=\mathbf{s}_{i,k}]}\tau(\mathrm{d}t_{-i}|t_i). \quad (1)$$

There are two important ways in which the third term in the above expression may be omitted. If for all $k = 1, \ldots, K$, the distribution $F_{\mathbf{s}_{i,k}}(\bullet|t_i)$ has no atoms and therefore, the tie-breaking rule (i.e., allocation rule a) is not important and, if the auctioneer keeps the objects in case of ties. That is, when $a_{i,k}(\circ) = 0$ whenever $b_{i,k} = \mathbf{s}_{i,k}$.

3. Bidding behavior

Our first result is a characterization of the payoff through its derivative with respect to the bid given by an integral expression (i.e., a kind of first fundamental theorem of calculus). For this, we will need the following assumption.

Condition 1. $u_{i,k} : T \times \mathbb{R}^{K(N+1)} \to \mathbb{R}$, $k = 0, 1, \ldots, K$ are absolutely continuous on $b_{i,k}$ and $\partial_{b_{i,k}} u_{i,k}$ is essentially bounded.[10]

Our main result is the following lemma.

Lemma 1 (Payoff characterization). *Assume Condition 1. Fix a b_i in the interior of B and profile of bidding functions $\mathbf{b}_{-i}$.[11] Define $b_{i,K+1} = \underline{b}_K$ and for $j = 1 \ldots, K$, let b_i^j denote the vector obtained from b_i by substituting the coordinate $b_{i,j}$ by $b_{i,j+1}$. Then for all $j = 1 \ldots, K$ the payoff of bidder i when bidding b_i can be expressed as*:

$$\Pi_i(t_i, b_i, \mathbf{b}_{-i}) = \Pi_i(t_i, b_i^j, \mathbf{b}_{-i}) + \int_{[b_{i,j+1}, b_{i,j})} \partial_{b_{i,j}}\Pi_i(t_i, (\beta, b_{i,-j}), \mathbf{b}_{-i})\,\mathrm{d}\beta$$
$$+ \int_{[b_{i,j+1}, b_{i,j})} E[u_{i,j}(\cdot)|t_i, \mathbf{s}_{i,k} = \beta]F^{\perp}_{\mathbf{s}_{i,j}}(\mathrm{d}\beta|t_i) + \sum_{k=1}^{K} E[a_{i,k}(\circ)u_{i,k}(\cdot)1_{[b_{i,k}=\mathbf{s}_{i,k}]}|t_i]$$

where $E[\cdot|t_i]$ is the expectation with respect to the measure $\tau(\cdot|t_i)$, and for almost all $b_{i,j}$:

$$\partial_{b_{i,j}}\Pi_i(t_i, (\beta, b_{i,-j}), \mathbf{b}_{-i}) = E[\partial_{b_{i,j}}u_{i,0}(\beta, \cdot)|t_i] + \sum_{k \neq j} E[\partial_{b_{i,j}}u_{i,k}(\beta, \cdot)1_{[b_{i,k}>\mathbf{s}_{i,k}]}|t_i] + E[\partial_{b_{i,j}}u_{i,j}(\beta, \cdot)1_{[\beta>\mathbf{s}_{i,j}]}|t_i]$$
$$+ E[u_{i,j}(\cdot)|t_i, \mathbf{s}_{i,j} = \beta]f_{\mathbf{s}_{i,j}}(\beta|t_i). \quad (2)$$

Proof. See Appendix A. □

The most important part of Lemma 1 is Eq. (2). When there is no tie with positive probability at b_i (i.e., $F_{\mathbf{s}_{i,k}}(-|t_i)$ has no atoms), $\partial_{b_{i,j}}\Pi_i(t_i, b_i, \mathbf{b}_{-i})$ is, for almost all $b_{i,j}$, the partial derivative of $\Pi_i(t_i, b_i, \mathbf{b}_{-i})$ (see Section 5). It is useful to observe that in Eq. (2), the first three terms on the right capture only the impact of changing bid $b_{i,j} = \beta$ in the payoff (payment) of each unit, while the last line captures the impact of such a change in the probability of winning the unit j. Note also the difference in the events in the second and the third line: $[b_{i,k} > \mathbf{s}_{i,k}]$ and $[\beta > \mathbf{s}_{i,j}]$.

The following corollary characterizes best response bids in an intuitive way. It says that under Condition 1, the optimum bid is such that the marginal cost of bidding is equal to the marginal utility from bidding. More formally, see the following corollary.

Corollary 1 (Basic principle of bidding). *Assume Condition 1. If $\Pi_i(\cdot)$ is differentiable in b_i at a bid profile which is optimal and in the interior of B, that is, at $b_i \in \operatorname{argmax}_{b \in B}\Pi_i(t_i, b, \mathbf{b}_{-i}) \cap \operatorname{int}B$, and there is no tie with positive probability at b_i (i.e., $F_{\mathbf{s}_{i,k}}(\cdot|t_i)$*

[10] Absolute continuity with respect to $b_{i,k}$ implies that $\partial_{b_{i,k}}u_{i,k}$ exists almost everywhere (with respect to Lebesgue measure) and

$$u_{i,k}(b_{i,j}, \cdot) - u_{i,k}(b_{i,j+1}, \cdot) = \int_{[b_{i,j+1}, b_{i,j})} \partial_{b_{i,j}}u_{i,0}(\beta, \cdot)\,\mathrm{d}\beta.$$

Essentially boundedness is used to invoke Lebesgue dominated convergence theorem.

[11] This interiority assumption does not allow for flat bids.

has no atoms), then for all j,

$$E[u_{i,j}(\cdot)|t_i, \mathbf{s}_{i,j} = b_{i,j}]f_{\mathbf{s}_{i,j}}(b_{i,j}|t_i) = E[-\partial_{b_{i,j}} u_{i,0}(b_{i,j}, \cdot)|t_i] + \sum_{k=1}^{K} E[-\partial_{b_{i,j}} u_{i,k}(b_{i,j}, \cdot)1_{[b_{i,k} > \mathbf{s}_{i,k}]}|t_i]. \tag{3}$$

Proof. If $F_{\mathbf{s}_{i,k}}(\cdot|t_i)$ has no atoms then $F^{\perp}_{\mathbf{s}_{i,k}}(\cdot|t_i) = 0$ almost everywhere. Therefore, by the payoff characterization lemma:

$$\Pi_i(t_i, b_i, \mathbf{b}_{-i}) = \Pi_i(t_i, b_i^j, \mathbf{b}_{-i}) + \int_{[b_{i,j-1}, b_{i,j})} \partial_{b_{i,j}} \Pi_i(t_i, (\beta, b_{i,-j}), \mathbf{b}_{-i})\, d\beta.$$

If $\Pi_i(t_i, b_i, \mathbf{b}_{-i})$ is differentiable at $b_i \in argmax_{b \in B} \Pi_i(t_i, b, \mathbf{b}_{-i})$ and is in the interior of B then

$$\partial_{b_{i,j}} \Pi_i(t_i, (b_{i,j}, b_{i,-j}), \mathbf{b}_{-i}) = 0.$$

This concludes the proof. □

Observe that $E[u_{i,j}(\cdot)|t_i, \mathbf{s}_{i,j} = b_{i,j}]f_{\mathbf{s}_{i,j}}(b_{i,j}|t_i)$ represents the marginal benefit of raising the bid in unit j, that is, the expected utility conditional to the event of a tie exactly for that unit. On the other hand, the terms in the second line of (3) represent the marginal cost of changing bid $b_{i,j}$, since the change of $b_{i,j}$ may affect the expected payment for all units one is winning. Note that this interpretation does not require separability in the monetary transfer (risk neutrality). This interpretation is useful for explaining bidding behavior in an intuitive way: the players bid to equalize the marginal benefit to the marginal cost of bidding.

The following corrollary will be used later to prove a monotone best-reply result.[12]

Corollary 2 (Payoff characterization as a line integral). *Assume Condition 1 and suppose for all i and k, $\partial_{b_{i,k}} \Pi_i$ exists and is continuous in b_i. Fix b_0, a profile of bidding functions $\mathbf{b}_{-i}$, two bids b_i^1 and b_i^2 in the interior of B, and a smooth curve $\alpha : [0, 1] \to B$ such that $\alpha(0) = b_i^0$ and $\alpha(1) = b_i^1$ then, for all $j = 1, \ldots, K$ the payoff of bidder i when bidding b_i can be expressed as*:

$$\Pi_i(t_i, b_i^1, \mathbf{b}_{-i}) = \Pi_i(t_i, b_i^0, \mathbf{b}_{-i}) + \int_{[0,1]} \nabla_{b_i} \Pi_i(t_i, \alpha(s), \mathbf{b}_{-i}) \cdot \alpha'(s)\, ds,$$

where $\nabla_{b_i} \Pi_i(t_i, \alpha(s), \mathbf{b}_{-i}) = (\partial_{b_{i,j}} \Pi_i(t_i, \alpha(s), \mathbf{b}_{-i}))_{j=1,\ldots,K}$.

Since we assume the existence and continuity of interim payoffs, this result follows immediately from the second fundamental theorem of calculus (see Apostol, 1967). The main point is that, under these conditions, we can explicitly write the derivative of interim expected payoffs. Theorem 1 of Krishna and Maenner (2001) shows that convexity or regular Lipschitzian condition are sufficient to guarantee the existence of a subgradient and, therefore, a representation of payoffs' as a line inegral. However, they do not calculate explicitly the subgradient.

3.1. Examples

The examples below show that Corollary 1 is a generalization of the necessary first-order conditions for the first and second-price auctions presented in Milgrom and Weber (1982), for the war of attrition and all-pay auctions presented in Krishna and Morgan (1997). The example on double auctions shows that the Basic Principle of Bidding is concise. Such an example is the application of Corollary 1 for double auctions and it presents a comparison with the equivalent expression obtained by Williams (1991).

Example 6 *(First price—single-object auction)*. When we restrict ourselves to the case of the first-price single object auction with risk neutrality: $K = 1$, $u_{i,0} = 0$ and $u_{i,1}(t, b) = v_{i,1}(t) - b_i$, then $\partial_{b_i} u_{i,1}(t, b) = -1$. The condition of Corollary 1 becomes:

$$b_i = E[v_{i,1}|t_i, \mathbf{s}_{i,1} = b_i] - \frac{F_{\mathbf{s}_{i,1}}(b_i|t_i)}{f_{\mathbf{s}_{i,1}}(b_i|t_i)}. \tag{4}$$

This (necessary) first-order condition provides a useful way to determine best-reply bids. Note that this expression admits non-monotonic bidding functions, contrary to Milgrom and Weber's model. It also encompasses asymmetries in valuations and distribution of types. Assuming affiliation and monotonic utilities, Milgrom and Weber (1982) can restrict themselves to the space of monotone symmetric bidding functions (i.e., $\mathbf{b}_i = \mathbf{b}$, for all $i \in N$). Thus,

$$\mathbf{s}_{i,1}(\mathbf{b}_{-i}(t_{-i})) = x \Leftrightarrow \max_{j \neq i} \mathbf{b}(t_j) = x \Leftrightarrow \max_{j \neq i} t_j = (\mathbf{b})^{-1}(x),$$

[12] This is a kind of second fundamental theorem of calculus for line integrals (see Apostol, 1967). By assuming the existence and continuity of interim payoffs, our result follows immediately from the second fundamental theorem of calculus. Our point is that, under these conditions, we can explicitly write the derivative of interim expected payoffs. Convexity or regular Lipschitzian condition in Theorem 1 of Krishna and Maenner guarantee the existence of a subgradient. We, however, assume the existence of such a derivative.

where in the last equation $(\mathbf{b})^{-1}$ stands for the inverse (generalized) of $\mathbf{b}$. This equation says that conditioning on $\mathbf{s}_{i,1} = b_i$ is the same to conditioning on $\max_{j \neq i} t_j = t_i$. Recall that $t_{(-i)} \equiv \max_{j \neq i} t_j$. Then $f_{\mathbf{s}_{i,1}}(s|t_i) = (f_{t_{(-i)}}(s|t_i)/\mathbf{b}'(s))$ and $F_{\mathbf{s}_{-i,1}}(s|t_i) = F_{t_{(-i)}}(s|t_i)$. With this, (4) becomes

$$\frac{d\mathbf{b}}{dt}(t_i) = (E[v_{i,1}|t_i, t_{(-i)} = t_i] - \mathbf{b}(t_i))\frac{f_{t_{-i}}(t_i|t_i)}{F_{t_{-i}}(t_i|t_i)}, \tag{5}$$

whose solution is shown to be an equilibrium under affiliation.

Example 7 *(Second price—single-object auction).* In the second price single object auction, Milgrom and Weber's model is equivalent to $K = 1$, $u_{i,1}(t, b) = v_{i,1}(t) - \mathbf{s}_{i,1}$ and $u_{i,0} = 0$. Then, $\partial_{b_i} u_{i,1}(t, b) = 0$ and the condition in Corollary 1 reduces to $E[v_{i,1} - b_{i,1}|t_i, \mathbf{s}_{i,1} = b_{i,1}]f_{\mathbf{s}_{i,1}}(b_{i,1}|t_i) = 0$ which can be simplified to

$$b_i = E[v_{i,1}|t_i, \mathbf{s}_{i,1} = b_{i,1}]. \tag{6}$$

Note that the marginal cost of bidding is zero for the second price auction. Again, with monotonicity and symmetry assumptions, Milgrom and Weber's expression for the equilibrium bid function can be obtained:

$$\mathbf{b}(t_i) = E[v_{i,1}|t_i, t_{(-i)} = t_i] \equiv \bar{v}(t_i, t_i).$$

Example 8 *(All pay—single-object auction).* Krishna and Morgan (1997) extend the method of Milgrom and Weber (1982) to the cases of war of attrition and all-pay auctions. In the all-pay auction, their model is equivalent to $u_{i,1}(t, b) = v_{i,1}(t) - \mathbf{s}_{i,1}$ and $u_{i,0}(t, b) = -b_{i,1}$. Then, $\partial_{b_{i,1}} u_{i,1}(t, b) = 0$ and $\partial_{b_{i,1}} u_{i,0}(t, b) = -1$. So, the condition in Corollary 1 reduces to

$$E[v_{i,1}|t_i, \mathbf{s}_{i,1} = b_{i,1}]f_{\mathbf{s}_{-i,1}}(b_{i,1}|t_i) = 1.$$

This means that the marginal cost of bidding is 1, just because the payment is made with certainty. The basic principle of bidding (that subsums to the above equation) says that the optimal bid is chosen to equalize the marginal benefit of bidding (the left hand side) to 1. Under the same hypothesis of monotonicity and symmetry, they find the following differential equation:

$$\frac{d\mathbf{b}}{dt}(t_i) = E[v_{i,1}|t_i, t_{(-i)} = t_i]f_{t_{(-i)}}(t_i|t_i),$$

whose solution they show to be an equilibrium under affiliation.

Example 9 *(War of attrition—single-object auction).* In the war of attrition, Krishna and Morgan (1997) model is equivalent to $u_{i,1}(t, b) = v_{i,1}(t) + b_{i,1} - \mathbf{s}_{i,1}$ and $u_{i,0}(t, b) = -b_{i,1}$. Then, $\partial_{b_i} u_{i,1}(t, b) = 1$ and $\partial_{b_{i,1}} u_{i,0}(t, b) = -1$. So, the condition in Corollary 1 reduces to

$$E[v_{i,1}|t_i, \mathbf{s}_{i,1} = b_{i,1}]f_{\mathbf{s}_{i,1}}(b_{i,1}|t_i) = 1 - F_{\mathbf{s}_{i,1}}(b_{i,1}|t_i).$$

Note that the marginal cost of bidding is less than 1, because the winner does not need to pay his bid. Again, with monotonicity and symmetry, they derive the equation

$$\frac{d\mathbf{b}}{dt}(t_i) = E[v_{i,1}|t_i, t_{(-i)} = t_i]\frac{1 - F_{t_{(-i)}}(t_i|t_i)}{f_{t_{(-i)}}(t_i|t_i)},$$

and the equilibrium is shown to exist under affiliation.

Example 10 *(Double auction).* In the analysis of a double auction with private values, risk neutrality, independent types and symmetry among buyers and sellers, Williams (1991) assumes that the payment is determined by the buyer's bid. So, it is optimum for the seller to bid his value. To analyze the behavior of the buyer i, Williams (1991) reaches the following expression:

$$\partial_{b_i} \Pi_i(v, \beta) = \left[n f_1(\beta) K_{n,m}(\mathbf{b}^{-1}(\beta), \beta) + (m-1)\frac{f_2(\mathbf{b}^{-1}(\beta))}{b'(\beta)} L_{n,m}(\mathbf{b}^{-1}(\beta), \beta) \right](v - \beta) - M_{n,m}(\mathbf{b}^{-1}(\beta), \beta), \tag{7}$$

where $\mathbf{b}$ denotes here the symmetric bidding function followed by all buyers, f_1 is the common density function of sellers, f_2 is the common density function of buyers, n is the number of sellers, m is the number of buyers and $M_{n,m}(\cdot,\cdot)$ is given by[13]

$$M_{n,m}(v,\beta) \equiv \sum_{\substack{i+j=m,\\ 0\le i\le m-1}} \binom{n}{j}\binom{m-1}{i} F_1(\beta)^j F_2(v)_i (1-F_1(\beta))^{n-j}\cdot(1-F_2(v))^{m-1-i};$$

$$K_{n,m}(v,\beta) \equiv \sum_{\substack{i+j=m-1,\\ 0\le i\le m-1}} \binom{n-1}{j}\binom{m-1}{i} F_1(\beta)^j F_2(v)_i \cdot(1-F_1(\beta))^{n-1-j}(1-F_2(v))^{m-1-i};$$

$$L_{n,m}(v,\beta) \equiv \sum_{\substack{i+j=m-1,\\ 0\le i\le m-1}} \binom{n}{j}\binom{m-2}{i} F_1(\beta)^j F_2(v)_i (1-F_1(\beta))^{n-j}\cdot(1-F_2(v))^{m-2-i}.$$

The expression (7) is just a special case of (3). To see this, observe that $F_{b_{(-i)}}(\beta)$, the probability that the threshold bid is less or equal to β, is given by the probability of the union of following disjoint events: there are i bids of buyers and j bids of sellers below or equal to β and $i+j=m$ (because the m th bid determines the threshold between winning and losing). Thus,

$$F_{\mathbf{s}_{i,1}}(\beta) = \sum_{\substack{i+j=m,\\ 0\le i\le m-1,\, 0\le j\le n}} \binom{n}{j}\binom{m-1}{i} F_1(\beta)^j \cdot F_2(\mathbf{b}^{-1}(\beta))_i (1-F_1(\beta))^{n-j}(1-F_2(\mathbf{b}^{-1}(\beta)))^{m-1-i},$$

which is equal to $M_{n,m}(\mathbf{b}^{-1}(\beta),\beta)$ above. Now, it is a matter of length but elementary derivation to confirm that

$$f_{\mathbf{s}_{i,1}}(\beta) = nf_1(\beta)K_{n,m}(\mathbf{b}^{-1}(\beta),\beta) + (m-1)\frac{f_2(\mathbf{b}^{-1}(\beta))}{b'(\beta)}L_{n,m}(\mathbf{b}^{-1}(\beta),\beta),$$

which concludes the proof of the claim.

Example 11 *(Multiple object discriminatory auction).* Let $u_{i,0}=0$, $u_{i,k}(t,b)=v_{i,k}(t)-b_{i,k}$. Then $\partial_{b_{i,j}}u_{i,k}(t,b)=0$ if $j\ne k$ and -1 if $j=k$. It is easy to show that the condition in Corollary 1 reduces to

$$b_{i,k} = E[v_{i,k}(t)|t_i, \mathbf{s}_{i,k}=b_{i,k}] - \frac{F_{\mathbf{s}_{i,k}}(b_{i,k}|t_i)}{f_{\mathbf{s}_{i,k}}(b_{i,k}|t_i)}.$$

Notice that demand reduction is an inmediate consequence of this characterization.

Example 12 *(Multiple object Vickrey auction).* Let $u_{i,0}=0$, $u_{i,k}(t,b)=v_{i,k}(t)-\mathbf{s}_{i,k}$. Then $\partial_{b_{i,j}}u_{i,k}(t,b)=0$. Therefore the condition in Corollary 1 reduces to

$$b_{i,k} = E[v_{i,k}(t)|t_i, \mathbf{s}_{i,k}=b_{i,k}].$$

Therefore in a general Vickrey Auction, an optimal bid is truthful.

Example 13 *(Uniform price auction).* Let $u_{i,0}=0$, $u_{i,k}(t,b)=v_{i,k}(t)-p$, where p is the payment, which is equal for all units and bidders. There are two common rules for the uniform price auction. One is the highest looser bid, which is the uniform price auction described by Krishna (2002). In this case, the payment is equal to the highest bid among those bids that do not receive the object. A variant is to put the payment equal to the lowest winning bid. We treat both below. Note that for any k,

$$\partial_{b_{i,j}}u_{i,k}(b_{i,j},\cdot) = -\partial_{b_{i,j}}p(b) = \begin{cases} -1, & \text{if } b_{i,j} \text{ determines the payment,}\\ 0, & \text{otherwise.}\end{cases}$$

In the case of the lowest winning bid, $b_{i,j}$ determines the payment in the event $s_{i,j}<b_{i,j}$ and $b_{i,j+1}<s_{i,j+1}$.[14] This event is contained in the event $[b_{i,k}>\mathbf{s}_{i,k}]$ if and only if $k\le j$. Thus, the first order condition becomes:

$$b_{i,j} = E[v_{i,j}(t)|t_i, \mathbf{s}_{i,j}=b_{i,j}] - j\frac{Pr[s_{i,j+1}>b_{i,j+1}, b_{i,j}>s_{i,j}]}{f_{\mathbf{s}_{i,j}}(b_{i,j}|t_i)}.$$

[13] To obtain $K_{n,m}(\cdot,\cdot)$ just substitute $n-1$ for n where it occurs in $M_{n,m}(\cdot,\cdot)$. To obtain $L_{n,m}(\cdot,\cdot)$, substitute $m-2$ for $m-1$ where it occurs in $M_{n,m}(\cdot,\cdot)$.

[14] We do not consider situations where two bids are equal.

In the case of the highest loosing bid, $b_{i,j}$ determines the payment if $b_{i,j} < s_{i,j}$, $b_{i,j-1} > s_{i,j-1}$ and $b_{i,j} > s_{i,j-1}$. Similarly,

$$b_{i,j} = E[v_{i,j}(t)|t_i, \mathbf{s}_{i,j} = b_{i,j}] - (j-1)\frac{Pr[s_{i,j+1} > b_{i,j+1}, b_{i,j} > s_{i,j}]}{f_{\mathbf{s}_{i,j}}(b_{i,j}|t_i)}.$$

Notice that demand reduction is an inmediate consequence of this characterization (see also Corollary 3 below).

4. Applications

Here we point out some potential applications and how our main result can be used to give a simple prove of some useful facts about auctions. Some of these results are new.

For all results below, we assume that the strategies $\mathbf{b}_{-i}$ of bidder i's opponents are such that the distribution of $\mathbf{s}_i$ is absolutely continuous with respect to the Lebesgue measure. Thus, the payoff is given only by the integral of its derivative.

4.1. Sufficient conditions for truthful bidding

It is known that second price auctions lead to bidding equal to the truthful expected value by the bidder. This can be easily seen from the first order condition (6) for this auction (see Example 7), reproduced below:

$$b_{i,1} = E[v_{i,1}|t_i, \mathbf{s}_{i,1} = b_{i,1}].$$

One may believe that the fact that truth-telling is optimal in the second-price auctions comes from the fact that the payment does not depend on the player's bid. This is not quite right, since a bidder's payment also does not depend on the player's bid in the third price auction, but the optimal bid in a third price auction is not the true player's valuation of the object.[15] Thus, the literature seems to lack a completely clear characterization of what leads to truthful bidding in auctions. The following results clarifies the two conditions that are sufficient for this.

Proposition 1. *Assume the conditions of* Corollary 1. (1) *If the bid $b_{i,j}$ never modifies the payment of any unit, then it is optimal for bidder i to bid $b_{i,j}$ such that*:

$$E[u_{i,j}(\cdot)|t_i, \mathbf{s}_{i,j} = b_{i,j}] = 0.$$

(2) *In addition to the previous condition, assume that $u_{i,j}(t,b) = v_{i,j}(t) - p_{i,j}(b)$, and that the payment $p(b)$ is $b_{i,j}$ in case of a relevant tie at $b_{i,j} = s_{i,k}$. Then the optimal bid is to bid the expected value of the unit*:

$$b_{i,j} = E[v_{i,j}(t)|t_i, \mathbf{s}_{i,j} = b_{i,j}].$$

Proof. (1) It is sufficient to examine the expression of $\partial_{b_{i,j}} \Pi_i(t_i, (\beta, b_{i,-j}), \mathbf{b}_{-i})$. (2) Observe that in the conditional event $\mathbf{s}_{i,j} = b_{i,j}$, the payment is $p(b) = b_{i,j}$. □

It is instructive to rephrase the above result in terms of the basic principle of bidding. The first condition requires that the marginal cost of bidding be zero. By the basic principle of bidding, one also has that the marginal benefit of bidding is zero. The second condition implies that the marginal benefit of bidding is the expected value of th object less the bid. From this, it follows that the bid is truthful. Some known results are immediate corollaries:

Corollary 3. *The first (highest) bid in the uniform price auction (with payment equal to the highest looser bid) is truthful.*[16]

Proof. The first bid cannot affect the payment of a winning bidder in a uniform auction, and thus the condition in (1) is satisfied. It is easy to see that the condition in (2) is also satisfied. □

Corollary 4. *The bids in the Vickrey auction are truthful.*

Proof. It is immediate to see that Vickrey auction satisfy the conditions in (1) and (2). □

It is useful to illustrate that the condition in (1) *is not* sufficient to get the result. The above mentioned case of the third price auction illustrates the point:

Example 14 *(Third price auction).* Consider the third-price auction, that is, the auction of a single object with $n \geq 3$ bidders, where the bidder with the highest bid wins and pays the third highest bid. Of course, the payment never depends on the winner's bid, that is, $\partial_{b_{i,1}} u_{i,1}(t,b) = 0$. Let us assume risk neutrality and private values, that is, $u_{i,1}(t,b) = t_i - p(t)$. Then, Proposition 1 (1) implies that: $t_i = E[p(b)|t_i, \mathbf{s}_{i,1} = b_{i,1}]$. However, condition (2) is not satisfied because this is strictly below

[15] We are grateful to Robert Marshall for making this observation. Example 14 was inspired by this.

[16] Menezes and Monteiro (2005) exhibit a uniform auction where truthful bidding is not equilibrium (see their Theorem 23, p. 131). Their example does not satisfy our assumptions because we assume away complementarities and flat demands, that is, the marginal value of an additional object should be nonincreasing of the objects need to be decreasing.

the bid $b_{i,1} : \mathbf{s}_{i,1}$ is the second highest bid and the expected payment (expected value of the third highest bid) is strictly below it. Thus, the optimal bid in a third price auction is above the own bidder's value.

The following example is also a useful illustration of the failure of the conditions of Proposition 1.

Example 15 *(Discriminatory highest others bid auction).* Consider a multi-unit auction with a standard allocation rule, that is, the highest K bids win the object. The payment of bidder i for unit j is given by $\max\{b_{l,k} : l \neq i, b_{l,k} \leq b_{i,j}\}$, that is, the payment is the highest defeated bid, not given by bidder i himself. This payment rule is interesting because is is easy to see that the payment for each unit dominates the payment for each unit in a standard Vickrey auction. In fact, in the standard Vickrey auction, the payment for unit j is just $\mathbf{s}_{i,j}$ which is in general below $\max\{b_{l,k} : l \neq i, b_{l,k} \leq b_{i,j}\}$. Therefore, if equilibrium in this auction were truthfull, the expected revenue of this auction would be above the Vickrey auction. Note also that condition (2) of Proposition 1 is satisfied: in the event of a tie $\mathbf{s}_{i,j} = b_{i,j}$, the payment is $\max\{b_{l,k} : l \neq i, b_{l,k} \leq b_{i,j}\} = \mathbf{s}_{i,j} = b_{i,j}$. However, condition (1) in Proposition 1 is not satisfied. This condition requires that each bidder's bid does not modifies his payments, but the payment for unit j increases if the set of bids $\{b_{l,k} : l \neq i, b_{l,k} \leq b_{i,j}\}$ changes with an increase of $b_{i,j}$.[17] Thus, bidding in this auction is not truthful, which is in line with Krishna and Perry (1998) or Krishna (2002) which have shown that the Vickrey auction gives the highest revenue among all truthful bidding mechanisms.

4.2. Sufficient conditions for increasing best reply

Let

$$V_i(\mathbf{b}_i, \mathbf{b}_{-i}) = \int \Pi_i(t_i, \mathbf{b}_i(t_i), \mathbf{b}_{-i})\,\mathrm{d}t_i$$

be the ex-ante payoff. We define the interim and the ex-ante best-reply correspondence, respectively, by

$$\Theta_i(t_i, \mathbf{b}_{-i}) \equiv \mathrm{argmax}_{\beta \in \mathcal{B}} \Pi_i(t_i, \beta, \mathbf{b}_{-i}),$$

and

$$\Gamma_i(\mathbf{b}_{-i}) \equiv \mathrm{argmax}_{\mathbf{b}_i \in \mathcal{L}^1([0,1],\mathcal{B})} V_i(\mathbf{b}_i, \mathbf{b}_{-i}).$$

We need the following definition.

Definition 2. Given a partial order $\succcurlyeq$ on T_i, we say that a function $g(t, b)$ is strictly increasing (non-decreasing) in t_i if $t_i^2 \succ t_i^1 (t_i^2 \succcurlyeq t_i^1)$ implies $g(t_i^2, t_{-i}, b) > (\geq) g(t_i^1, t_{-i}, b)$ for all t_{-i}, b.[18]

Let $\geq$ denote the coordinate-wise partial order in B, that is: $b_i^1 \geq b_i^0$ if $b_{i,j}^1 \geq b_{i,j}^0$ for all $j = 1, \ldots, K$. We write $b_i^1 > b_i^0$ if $b_i^1 \geq b_i^0$ and $b_i^1 \neq b_i^0$.

Proposition 2. *Assume we are under the conditions of* Corrollary 2 *and that types are independent. Let $\succcurlyeq$ be a partial order on T_i. For all $k, j = 1, \ldots, K$, assume that $u_{i,k}(t, b)$ is absolutely continuous in t and b, and strictly increasing in t_i; and $\partial_{b_{i,j}} u_{i,k}(t, b)$ is non-decreasing in t_i (except, possibly in a set of null measure). Then the following holds*:

1. *For each t_i, $\Theta_i(t_i, \mathbf{b}_{-i})$ is non-empty.*
2. *Consider two types $t_i^1, t_i^2, t_i^2 \succ t_i^1$, and best reply bids for them, that is, $b_i^1 \in \Theta_i(t_i^1, \mathbf{b}_{-i})$, $b_i^2 \in \Theta_i(t_i^2, \mathbf{b}_{-i})$ and assume that these bids imply different probabilities of winning, i.e.*,

$$Pr(\{t_{-i} : \exists k \quad \text{such that} \quad a_{i,k}(b_i^1, \mathbf{b}_{-i}(t_{-i})) \neq a_{i,k}(b_i^2, \mathbf{b}_{-i}(t_{-i}))\}) > 0.$$

Then $\sim (b_i^1 > b_i^2)$.

Proof. See Appendix A. □

The unidimensional version of the above theorem was used by Araujo and de Castro (2008) to prove equilibrium existence in single-unit auctions. The main role of this result in their equilibrium proof is to restrict the set of strategies to a compact set (the set of non-decreasing functions). Restricted to this strategy set, they obtained approximated equilibria of perturbed games, used compactness to obtain a converging subsequence and proved that the limit is equilibrium of the original auction.

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17 There is also another way to see the same problem: Condition 1 assumed in Corollary 1 fails to be satisfied. This requires the utility functions to be absolutely continuous, which implies that the utility functions are the integral of their (almost everywhere) derivative. But this is clearly false here (see footnote 10).

18 Given the partial order $\succcurlyeq$, we write $x \succ y$ if $x \succcurlyeq y$ but $\sim (y \succcurlyeq x)$.

Maybe the above theorem could be equally useful in obtaining new equilibrium existence results for multiunit auctions, but such results are out of the scope of this paper.

4.3. Identification of multi-unit auctions

In economic theory the problem of identification is whether a observable data, conditional to some theory, uniquely identify the unobservable primitives. In auctions, we condition to Bayesian–Nash equilibrium theory; observables are typically all agents bids, equilibrium prices, agents identities and/or some covariates (such as interest rates, prices in secondary markets, etc.). Primitives of any auction include the set players, the information structure, the valuation or preference structure and players objectives. Therefore, the question is if different primitives for example, different preferences, have, under the hypothesis that players play equilibrium strategies (Bayesian–Nash), undestinguishable observable outcomes such as equilibrium prices. This problem is central for several normative issues since, for example, preferences, are the basis for welfare comparisons among players. Moreover, in auction theory primitives are central to market design, optimality (revenue for the auctioner), efficiency, reserve prices, etc. By now there is large literature on identification of unitary auctions (see Paarsh and Hong, 2006 for an overview of the structural econometrics of auctions) but only recently the case of multiple unit demand auctions has been the focus of attention. The problem is of interest in the applied literature because many important markets rely on auction mechanisms to allocate goods or services and these are naturally modeled as markets for the allocation of multiple units. Prominent examples are the markets for treasury auctions, the demand and supply of electricity and the markets of emissions permits of carbon dioxide.[19] Also, there has been an increasing effort in studying identification results within a nonparametric setting (see Athey and Haile, 2002, 2007). Nonparametric identification is important because it gives valid interpretations of estimates on finite samples. It also sheds light on what results of the theory are trully essential as opposed to those that are just artifacts of a particular parametrization.

With very few exceptions (for example McAdams, 2007 that we comment below), most applied work rely on Wilson (1979) share model. One of the salient features of this approach is the use of continuously differentiable bid functions for a divisible good. In real markets, however, the particular institutional settings in which some of these auctions are carried make both assumptions, divisible goods and the continuosly differentiable bid functions unattainable. For example, in England and Wales spot electricity market generators make three bids out of their supply function.[20] In Texas balancing electricity market, bidders are allowed to offer up to 40 points of their supply curve and in the Colombian spot electricity market, generators make price offers for generating a fixed amount of energy per generating unit. This examples question the continuosly differentiable bid assumption. Now, almost always there are minimun incremental quantities and price offers. For example, in England and Whales spot electricity markets, fractions of pennies are not allowed and in treasury auctions there is usually a minimun denomination for bonds. Hence, it is important to study identification when bids are not continuous or quantities and/or bids are discrete. Below we provide nonparametric identification results, in a nondifferentiable, discrete setting for the two most important multi-unit auctions, the discriminatory auction and the uniform auction. More precisely, goods are not divisible but bids are divisible (therefore, we have not taking in considerations all sources of discreetness in real markets). To the extent of our knowledge, our result for the interdependent values discrete case discriminatory multi-unit auction is new. Below we make detailed comments on some related existing results in the literature (in particular, Hortacsu, 2002; Hortacsu and Puller, 2008).

4.3.1. Assumptions

Assume that all players are risk-neutral and that agents beliefs are fixed across auctions. Furthermore, assume that each auction (information realization of signals) is independent.[21] Our behavioral assumption is that in each auction players follow best response strategies as characterized in this article. An important issue arises weather best response strategies satisfy all our assumptions (for example, that the best response is interior to the set of feasible bids) or if they are unique.[22] We set aside these important issues by assuming that the same best response strategy is played in every auction for every player and that Corollary 1 applies. Notice that our characterization results hold almost everywhere but in fact, this is all we need for identification.

We assume the number of bidders is fixed in every auction and that the econometrician observes all players bids and identities. In the case of symmetric bidders, only players bids are assumed to be observed.

4.3.2. Results

Recall Example 13 where we derived the first order conditions for the multi-unit uniform auction. Consider the case in which agents pay the lowest winning bid (the other case is similar).

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[19] See Hortacsu (2002) for treasury auctions, Hortacsu and Puller (2007) and Wolak (2006) for electricity markets and Ellerman et al. (2000) for tradable emission permits of carbon dioxide.

[20] See Wolfram (1998).

[21] In the auction literature on identification these assumptions are standard (see Athey and Haile, 2007).

[22] Nondifferentiable bids and uniqueness are important issues addressed by McAdams (2007).

Proposition 3. *Consider the uniform price auction* (*see* Example 13). *If values are private and best reply strategies are not flat, the marginal utility of an additional unit is nonparametrically identified from agents bid. Formally*,

$$v_{i,j}(t_i) = b_{i,j} + j\frac{Pr[s_{i,j+1} > b_{i,j+1}, b_{i,j} > s_{i,j}]}{f_{\mathbf{s},j}(b_{i,j}|t_i)}. \tag{8}$$

If values are interdependent, we can only identify the conditional expectation $E[v_{i,j}(t)|t_i, \mathbf{s}_{i,j} = b_{i,j}]$. *Formally*,

$$E[v_{i,j}(t)|t_i, \mathbf{s}_{i,j} = b_{i,j}] = b_{i,j} + j\frac{Pr[s_{i,j+1} > b_{i,j+1}, b_{i,j} > s_{i,j}]}{f_{\mathbf{s},j}(b_{i,j}|t_i)}. \tag{9}$$

Proof. To the extent that all agents' bids are observable and one is able to nonparametrically estimate the second term from the right hand side, identification follows.[23] □

Eq. (8) is analogous to Eq. (2) in Hortacsu and Puller (2008). The main difference is that they assume a divisible good and continuously differentiable bid functions. Although we do not address explicitly the role of discontinuous bid functions, it is well known that some of the surprising results in share auctions (e.g. low equilibrium closing prices) are due to the special nature of continuously differentiable bid functions (see Kremer and Nyborg, 2004).

Proposition 4. *Consider the discriminatory multi-unit auction* (*see* Example 11). *Then, if values are private, the marginal utility of an additional unit is nonparametrically identified from agents bid. Formally*,

$$v_{i,j}(t_i) = b_{i,j} + \frac{F_{\mathbf{s}_{i,j}}(b_{i,j}|t_i)}{f_{\mathbf{s}_{i,j}}(b_{i,j}|t_i)}. \tag{10}$$

If values are interdependent, we can only identify the conditional expectation $E[v_{i,k}(t)|t_i, \mathbf{s}_{i,k} = b_{i,k}]$. *Formally*,

$$E[v_{i,k}(t)|t_i, \mathbf{s}_{i,k} = b_{i,k}] = b_{i,k} + \frac{F_{\mathbf{s}_{i,k}}(b_{i,k}|t_i)}{f_{\mathbf{s}_{i,k}}(b_{i,k}|t_i)}. \tag{11}$$

Proof. The same argument as before. □

Eq. (10) is analogous to Eqs. (2) and (8) in Hortacsu (2002). The first one assumes divisible goods and continuously differentiable bids, the second assumes a divisible good but restricts bids to be in a discrete price grid.

In the case of interdependent values, notice the difference between Eq. (12) in Hortacsu and Eq. (11) above. In our case we are able to identify the conditional expected value of valuation while in Hortacsu, these can only be done by imposing additional assumptions (these are basically that, in his first order conditions, one is able to solve for the differential equation in the expected value of valuation).

Finally a few comments on McAdams (2005). As opposed to most of the literature on identification in multi-unit auctions, this paper characterizes the full set of valuations distributions that could be consistent with market data. As we pointed out before, first order conditions provide only necessary conditions for a particular distribution of observable bids to be consistent with optimal behavior. In case of multiple best reply strategies, we assumed that observables where sample observations from the same best reply strategy. Clearly, to assume that observables are a sample observation of the same best reply strategy is an important limitation of our identification strategy in multiple-unit auctions where, even under symmetry assumptions, there can be multiplicity of equilibria. McAdams avoids these limitations by relying on nondifferentiable methods for characterizing optimal behavior. His paper generalizes Propositions 2 and 3 in the case of private values to the case of none interior bid functions. To the extent of our knowledge, our Proposition 4 with interdependent values is new.

5. Conclusion

Differential methods have a widespread use in auction theory. In this paper, we show that the standard approach of taking the first order condition to characterize optimal bidding can be extended to multi-unit auctions in a general setting. The main limitation of this approach is that it requires best reply interior bids. This can be an important limitation in some applications. Another limitation is the fact that bidders place different bids for all K units. In many real world auctions, there is a limit to the number of different bids that a bidder can place.

However, the differentiable approach provide a simple characterization that has an intuitive explanation. Other potential applications are: (1) Using the sufficient conditions for truthful bidding, design auctions that are close to truthful, but do not have the problems of Vickrey auctions. (2) It is likely that the monotone best reply result can be used in proving equilibrium existence results. (3) Characterization of sufficient conditions for best reply strategies and testable restrictions. (4) Characterize efficient multi-unit auctions and optimal behavior in other nonstandard multi-unit auction such as the Spanish or the k th - average price auction.

[23] This is the main idea put forward in nonparemetric identification results. Athey and Haile (2007) attribute this idea to Guerre et al. (2000) and earlier papers of the same authors.

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Appendix A. Proofs

Proof of the payoff characterization lemma. The proof follows the demonstration of the Leibniz rule. The main point is the use of a well known theorem on the derivatives of measures and its integral expression. The theorem we use is in Rudin (1966).

Recall the expression for agents expected payoff:

$$\Pi_i(t_i, b_i, \mathbf{b}_{-i}) = \int_{T_{-i}} u_{i,0}(\cdot)\tau(\mathrm{d}t_{-i}|t_i) + \sum_{k=1}^{K}\int_{T_{-i}} u_{i,k}(\cdot)1_{[b_{i,k}>\mathbf{s}_{i,k}]}\tau(\mathrm{d}t_{-i}|t_i) + \sum_{k=1}^{K}\int_{T_{-i}} a_{i,k}(\circ)u_{i,k}(\cdot)1_{[b_{i,k}=\mathbf{s}_{i,k}]}\tau(\mathrm{d}t_{-i}|t_i). \quad (12)$$

Fix j, and consider each term above separately.

1. The first one has a derivative with respect to $b_{i,j}$ almost everywhere and is equal to $E[\partial_{b_{i,j}} u_{i,0}(\cdot)|t_i]$. Also,

$$E[u_{i,0}(\cdot)|t_i] = \int_{[b_{i,j+1}, b_{i,j})} E[\partial_{b_{i,j}} u_{i,0}(\beta, \cdot)|t_i]\,\mathrm{d}\beta + c_0,$$

   where c_0 is a constant.
2. If the distribution $F_{\mathbf{s}_{i,k}}(-|t_i)$ has no atoms, the third term is equal to zero and its derivative exists and it's zero.
3. Now consider the second term, $\int_{T_{-i}} u_{i,k}(\cdot)1_{[b_{i,k}>\mathbf{s}_{i,k}]}\tau(\mathrm{d}t_{-i}|t_i)$. There are two cases, $j \neq k$ and $j = k$. In the first case ($j \neq k$), let $a^n \to (b_{i,j})^+$ (i.e., $a^n > b_{i,j}$; the case $a^n \to (b_{i,k})^-$ is analogous). We have

$$\int_{T_{-i}} u_{i,k}(t_i, t_{-i}, (a^n, b_{i,-j}), \mathbf{b}_{-i}(t_{-i}))1_{[b_{i,k}>\mathbf{s}_{i,k}]}\tau(\mathrm{d}t_{-i}|t_i) - \int_{T_{-i}} u_{i,k}(\cdot)1_{[b_{i,k}>\mathbf{s}_{i,k}]}\tau(\mathrm{d}t_{-i}|t_i)$$
$$= \int_{T_{-i}} (u_{i,k}(t_i, t_{-i}, (a^n, b_{i,-j}), \mathbf{b}_{-i}(t_{-i})) - u_{i,k}(\cdot))1_{[b_{i,k}>\mathbf{s}_{i,k}]}\tau(\mathrm{d}t_{-i}|t_i).$$

   Since u_i has bounded derivative with respect to almost all $b_{i,j}$,

$$\lim_{a^n\to(b_{i,j})^+} \frac{u_{i,k}(t_i, t_{-i}, (a^n, b_{i,-j}), \mathbf{b}_{-i}(t_{-i})) - u_{i,k}(\cdot)}{a^n - b_{i,j}} = \partial_{b_{i,j}} u_{i,k}(\cdot),$$

   for almost all b_i. By Lebesgue dominated convergence theorem,

$$\lim_{a^n\to(b_i)^+} \int_{T_{-i}} \frac{u_{i,k}(t_i, t_{-i}, (a^n, b_{i,-j}), \mathbf{b}_{-i}(t_{-i})) - u_{i,k}(\cdot)}{a^n - b_{i,j}} 1_{[b_{i,k}>\mathbf{s}_{i,k}]}\tau(\mathrm{d}t_{-i}|t_i)$$

   exists and it is equal to $E[\partial_{b_{i,j}} u_{i,k}(\cdot)1_{[b_{i,k}>\mathbf{s}_{i,k}]}|t_i]$. Also,

$$E[u_{i,k}(\cdot)1_{[b_{i,k}>\mathbf{s}_{i,k}]}|t_i] = c_1 + \int_{[b_{0,j}, b_{i,j})} E[\partial_{b_{i,j}} u_{i,k}(\beta, \cdot)1_{[b_{i,k}>\mathbf{s}_{i,k}]}|t_i],$$

where c_1 is a constant. From now on, we will omit the constant terms, since it is clear that they sum up to $\Pi_i(t_i, b_i^j, \mathbf{b}_{-i})$.

In the second case ($j = k$), let $a^n \to (b_{i,k})^+$ (i.e., $a^n > b_{i,k}$; the case $a^n \to (b_{i,k})^-$ is analogous). Then,

$$\int_{T_{-i}} u_{i,k}(t_i, t_{-i}, (a^n, b_i, k), \mathbf{b}_{-i}(t_{-i}))1_{[a^n > \mathbf{s}_{i,k}]}\tau(\mathrm{d}t_{-i}|t_i) - \int_{T_{-i}} (\cdot)1_{[b_{i,k} > \mathbf{s}_{i,k}]}$$
$$= \int_{T_{-i}} (u_{i,k}(t_i, t_{-i}, (a^n, b_{i,-k}), \mathbf{b}_{-i}(t_{-i})) - u_{i,k}(\cdot))1_{[a^n > \mathbf{s}_{i,k}]}\tau(\mathrm{d}t_{-i|}t_i) + \int_{T_{-i}} u_{i,k}(\cdot)(1_{[a^n > \mathbf{s}_{i,k}]} - 1_{[b_{i,k}] > \mathbf{s}_{i,k}})\tau(\mathrm{d}t_{-i}|t_i)$$
$$= \int_{T_{-i}} (u_{i,k}(t_i, t_{-i}, (a^n, b_{i,-k}), \mathbf{b}_{-i}(t_{-i})) - u_{i,k}(\cdot))1_{[a^n > \mathbf{s}_{i,k}]}\tau(\mathrm{d}t_{-i}|t_i) + \int_{T_{-i}} u_{i,k}(\cdot)1_{[a^n > \mathbf{s}_{i,k} \geq b_{i,k}]}\tau(\mathrm{d}t_{-i}|t_i).$$

Since u_i has bounded derivative with respect to almost all $b_{i,k}$,

$$\lim_{a^n \to (b_{i,k})^+} \frac{u_{i,k}(t_i, t_{-i}, (a^n, b_{i,-k}), \mathbf{b}_{-i}(t_{-i})) - u_{i,k}(\cdot)}{a^n - b_{i,k}} = \partial_{b_{i,k}} u_{i,k}(\cdot),$$

for almost all $b_{i,k}$. Also, $1_{[a^n > \mathbf{s}_{i,k}]} \to 1_{[b_{i,k} > \mathbf{s}_{i,k}]}$. These imply that:

$$\lim_{a^n \to (b_{i,k})^+} \frac{u_{i,k}(t_i, t_{-i}, (a^n, b_{i,-k}), \mathbf{b}_{-i}(t_{-i})) - u_{i,k}(\cdot)}{a^n - b_{i,k}} 1_{[a^n > \mathbf{s}_{i,k}]} = \partial_{b_{i,k}} u_{i,k}(\cdot)1_{[b_{i,k} > \mathbf{s}_{i,k}]},$$

for almost all $b_{i,k}$ and these functions are (almost everywhere) bounded. Again by Lebesgue dominated convergence theorem, the limit

$$\lim_{a^n \to (b_{i,k})^+} \int_{T_{-i}} \frac{(u_{i,k}(t_i, t_{-i}, (a^n, b_{i,-k}), \mathbf{b}_{-i}(t_{-i})) - u_{i,k}(\cdot))}{a^n - b_{i,k}} 1_{[a^n > \mathbf{s}_{i,k}]}\tau(\mathrm{d}t_{-i}|t_i)$$

exists and it is equal to $E[\partial_{b_{i,k}} u_{i,k}(\cdot)1_{[b_{i,k} > \mathbf{s}_{i,k}]}|t_i]$. Also,

$$E[u_{i,k}(\cdot)1_{[b_{i,k} > \mathbf{s}_{i,k}]}|t_i] = \int_{[b_{0,k}, b_{i,k})} E[\partial_{b_{i,k}} u_{i,k}(t_i, t_{-i}, (\beta, b_{i,-k}), \mathbf{b}_{-i}(t_{-i}))1_{[\beta > \mathbf{s}_{i,k}]}|t_i]\,\mathrm{d}\beta.$$

Now, we want to determine:

$$\lim_{a^n \to (b_{i,k})^+} \frac{1}{a^n - b_{i,k}} \int_{T_{-i}} u_{i,k}(\cdot)1_{[a^n > \mathbf{s}_{i,k} \geq b_{i,k}]}\tau(\mathrm{d}t_{-i}|t_i).$$

For each each $t_i \in T_i$ and b_i fixed, define the signed measure ρ over $\mathbb{R}$ by[24]

$$\rho(V; t_i, b_i, \mathbf{b}_{-i}) \equiv \int_{T_{-i}} u_{i,k}(\cdot)1_{[\mathbf{s}_{i,k} \in V]}\tau(\mathrm{d}t_{-i}|t_i).$$

[24] On a σ-field this is synonymous of a countably additive set function.

Then,

$$\lim_{a^n \to (b_{i,k})^+} \frac{1}{a^n - b_{i,k}} \int_{T_{-i}} u_{i,k}(\cdot) 1_{[a^n > \mathbf{s}_{i,k} \geq b_{i,k}]} \tau(\mathrm{d}t_{-i}|t_i) = \lim_{a^n \to (b_{i,k})^+} \frac{1}{a^n - b_{i,k}} \rho([b_{i,k}, a^n); t_i, b_i, \mathbf{b}_{-i})$$
$$= \lim_{a^n \to (b_{i,k})^+} \frac{\rho([b_{i,k}, a^n); t_i, b_i, \mathbf{b}_{-i})}{m([b_{i,k}, a^n))},$$

where m is Lebesgue measure over $\mathbb{R}$. By Theorem 8.6 of Rudin (1966) this limit exists m-almost everywhere in $b_{i,k}$ and we call it $D\rho(t_i, b_i, \mathbf{b}_{-i})$. Also, $D\rho$ coincides almost everywhere with the Radon–Nikodym derivative $(\mathrm{d}\rho/\mathrm{d}m)(t_i, b_i, \mathbf{b}_{-i})$. Therefore,

$$\rho(V; t_i, b_i, \mathbf{b}_{-i}) = \int_V \frac{\mathrm{d}\rho}{\mathrm{d}m}(t_i, b_i, \mathbf{b}_{-i})\,\mathrm{d}m + \rho^{\perp}(V; t_i, b_i, \mathbf{b}_{-i}),$$

where $\rho^{\perp}$ denotes the singular part of ρ, and it has the property

$$\lim_{a^n \to (b_{i,k})^+} \frac{\rho^{\perp}([b_{i,k}, a^n); t_i, b_i, \mathbf{b}_{-i})}{m([b_{i,k}, a^n))} = 0$$

by the same theorem.

It is easy to see that ρ is absolutely continuous with respect to the distribution $F_{\mathbf{s}_{i,k}}(\cdot|t_i)$. The Radon–Nikodym Theorem guarantees the existence of the Radon–Nikodym derivative of ρ with respect to the distribution of $F_{\mathbf{s}_{i,k}}(\cdot|t_i)$, which we denote by g. Therefore, g is such that

$$\rho(V; t_i, b_i, \mathbf{b}_{-i}) = \int_V g(\beta) F_{\mathbf{s}_{i,k}}(\mathrm{d}\beta|t_i) = \int_V g(\beta) f_{\mathbf{s}_{i,k}}(\beta|t_i)\,\mathrm{d}\beta$$

and by definition:

$$\rho(V; t_i, b_i, \mathbf{b}_{-i}) \equiv \int_{T_{-i}} u_{i,k}(\cdot) 1_{[\mathbf{s}_{i,k} \in V]} \tau(\mathrm{d}t_{-i}|t_i) = E[u_{i,k}(\cdot) 1_{[\mathbf{s}_{i,k} \in V]}|t_i] = \int_{[b_{0,k},\infty)} E[u_{i,k}(\cdot) 1_{[\mathbf{s}_{i,k} \in V]}|t_i, \mathbf{s}_{i,k} = \beta] F_{\mathbf{s}_{i,k}}(\mathrm{d}\beta|t_i)$$
$$= \int_V E[u_{i,k}(\cdot)|t_i, \mathbf{s}_{i,k} = \beta] F_{\mathbf{s}_{i,k}}(\mathrm{d}\beta|t_i) = \int_V E[u_{i,k}(\cdot)|t_i, \mathbf{s}_{i,k} = \beta] f_{\mathbf{s}_{i,k}}(\beta|t_i)\,\mathrm{d}\beta.$$

Therefore, by the unicity of the Radom Nikodyn derivative of ρ with respect to Lebessgue measure m, we have that:

$$g(\beta) = E[u_{i,k}(\cdot)|\mathbf{s}_{i,k} = \beta] f_{\mathbf{s}_{i,k}}(\beta|t_i),$$

m-almost everywhere in β.

Thus,

$$\partial_{b_{i,j}}\Pi_i(t_i,(\beta,b_{i,-j}),\mathbf{b}_{-i}) = E[\partial_{b_{i,j}}u_{i,0}(\beta,\cdot)|t_i]\sum_{k\neq j}^{K}E[\partial_{b_{i,j}}u_{i,k}(\beta,\cdot)1_{[b_{i,k}>\mathbf{s}_{i,k}]}|t_i]+E[\partial_{b_{i,j}}u_{i,j}(\beta,\cdot)1_{[\beta>\mathbf{s}_{i,k}]}|t_i]E[u_{i,j}(\cdot)|t_i,\mathbf{s}_{i,j}$$
$$= \beta]f_{\mathbf{s}_{i,k}}(\beta|t_i).$$

Finally, by the Lebesgue dominated convergence theorem,

$$\Pi_i(t_i,b_i,\mathbf{b}_{-i}) = \int_{[b_{i,j-1},b_{i,j})}\partial_{b_{i,j}}\Pi_i(t_i,(\beta,b_{i,-j}),\mathbf{b}_{-i})\,\mathrm{d}\beta+\int_{[b_{i,j-1},b_{i,j})}E[u_{i,j}(\cdot)|t_i,\mathbf{s}_{i,k}$$
$$= \beta]F^{\perp}_{\mathbf{s}_{i,j}}(\beta|t_i)\sum_{k=1}^{K}\int_{T_{-i}}a_{i,k}(\circ)u_{i,k}(\cdot)1_{[b_{i,k}=\mathbf{s}_{i,k}]}\tau(\mathrm{d}t_{-i}|t_i).$$

This concludes the proof. □

Proof of Proposition 2.

(i) Since B is compact and $\Pi_i(t_i,\cdot,\mathbf{b}_{-i})$ is continuous if $F_{\mathbf{b}_{-i}}(\cdot)$ is absolutely continuous, the conclusion is immediate.
(ii) We will make use of the expression:

$$\partial_{b_{i,j}}\Pi_i(t_i,(\beta,b_{i,-j}),\mathbf{b}_{-i}) = E[\partial_{b_{i,j}}u_{i,0}(\beta,\cdot)|t_i]+\sum_{k\neq j}E[\partial_{b_{i,j}}u_{i,k}(\beta,\cdot)1_{[b_{i,k}>\mathbf{s}_{i,k}]}|t_i]$$
$$+E[\partial_{b_{i,j}}u_{i,j}(\beta,\cdot)1_{[\beta>\mathbf{s}_{i,j}]}|t_i]+E[u_{i,j}(\cdot)|t_i,\mathbf{s}_{i,j}=\beta]f_{\mathbf{s}_{i,j}}(\beta|t_i).$$

Since $b_i^1>b_i^2$, we can choose a curve $\alpha:[0,1]\to B$, such that $\alpha(0)=b_i^2$, $\alpha(1)=b_i^1$ and

$$\alpha_j'(s)\geq 0,\ \forall j,\ s\in[0,1]\quad\text{and}\quad \exists j\quad\text{such that}\quad \alpha_j'(s)>0,\ \forall s\in[0,1]. \tag{13}$$

By assumption, $\partial_{b_{i,j}}u_{i,k}(t_i^0,\cdot)\leq\partial_{b_{i,j}}u_{i,k}(t_i^1,\cdot)$ for all k. Thus,

$$E[\partial_{b_{i,j}}u_{i,k}(t_i^0,\cdot)1_{[\alpha_k(s)>\mathbf{s}_{i,k}]}]\leq E[\partial_{b_{i,j}}u_{i,k}(t_i^1,\cdot)1_{[\alpha_k(s)>\mathbf{s}_{i,k}]}]. \tag{14}$$

Since $[0,1]^{n-1}$ and $\mathcal{B}^n$ are compact and u_i is (absolutely) continuous, there exists $\delta>0$ such that $u_{i,k}(t_i^1,t_{-i},b)+2\delta<u_{i,k}(t_i^2,t_{-i},b)$ for all $t_{-i}\in[0,1]^{n-1}$, all $b\in B^n$ and all k. For fixed bid $\beta\in B$ and j, define the functions

$$g^1(t_{-i})=u_{i,j}(t_i^1,t_{-i},\beta,\mathbf{b}_{-i}(t_{-i}))\quad\text{and}\quad g^2(t_{-i})=u_{i,j}(t_i^2,t_{-i},\beta,\mathbf{b}_{-i}(t_{-i})).$$

Then, $g^1(t_{-i})+2\delta<g^2(t_{-i})$. By the positivity of conditional expectations,[25]

$$E[g^2-g^1-2\delta|\mathbf{s}_{i,j}=\beta]\geq 0.$$

Thus, from the independence of types, we conclude that

$$E[u_{i,j}(t_i^1,\cdot)|t_i,\mathbf{s}_{i,j}=\beta]+\delta<E[u_{i,j}(t_i^2,\cdot)|t_i,\mathbf{s}_{i,j}=\beta]. \tag{15}$$

Then, (14) and (15), and the expression of $\partial_{b_i}\Pi_i(t_i,\beta,\mathbf{b}_{-i})$ given by the characterization lemma imply that for almost all β,

$$\nabla_{b_i}\Pi_i(t_i^2,\alpha(s),\mathbf{b}_{-i})>\nabla_{b_i}\Pi_i(t_i^1,\alpha(s),\mathbf{b}_{-i})+\delta f_{\mathbf{s}}(\alpha(s)), \tag{16}$$

[25] See, for instance, Kallenberg (2002), Theorem 6.1, p. 104.

where $f_{\mathbf{s}}(\alpha(s))$ denotes the vector $(f_{\mathbf{s}_{i,1}}(\alpha_1(s)),\ldots,f_{\mathbf{s}_{i,K}}(\alpha_K(s)))$. The assumption on the distribution implied by $\mathbf{b}_{-i}$ allow to write the difference $\Pi_i(t_i^2,b_i^1,\mathbf{b}_{-i})-\Pi_i(t_i^2,b_i^2,\mathbf{b}_{-i})$ as an integral:

$$\Pi_i(t_i^2,b_i^1,\mathbf{b}_{-i})-\Pi_i(t_i^2,b_i^2,\mathbf{b}_{-i}) = \int_{[0,1]} \nabla_{b_i}\Pi_i(t_i^2,\alpha(s),\mathbf{b}_{-i})\cdot\alpha'(s)\,\mathrm{d}s > \int_{[0,1]} \nabla_{b_i}\Pi_i(t_i^1,\alpha(s),\mathbf{b}_{-i})\cdot\alpha'(s)\,\mathrm{d}s$$
$$+\delta\sum_j\int_{[0,1]} f_{\mathbf{s}_{i,j}}(\alpha_j(s))\alpha_j'(s)\,\mathrm{d}s \geq \delta\sum_j\int_{[0,1]} f_{\mathbf{s}_{i,j}}(\alpha_j(s))\alpha_j'(s)\,\mathrm{d}s \geq 0,$$

where the first inequality comes from (13) and (16); the second comes from the fact that $b_i^1\in\Theta_i(t_i^1,\mathbf{b}_{-i})$, that is,

$$\Pi_i(t_i^1,b_i^1,\mathbf{b}_{-i})-\Pi_i(t_i^1,b_i^2,\mathbf{b}_{-i}) = \int_{[0,1]} \nabla_{b_i}\Pi_i(t_i^1,\alpha(s),\mathbf{b}_{-i})\cdot\alpha'(s)\,\mathrm{d}s \geq 0;$$

and the third comes from $\alpha_j'(s)\geq 0$ for all j. Now, this implies that $\Pi_i(t_i^2,b_i^1,\mathbf{b}_{-i}) > \Pi_i(t_i^2,b_i^2,\mathbf{b}_{-i})$, which contradicts the fact that $b_i^2\in\Theta_i(t_i^2,\mathbf{b}_{-i})$. □

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