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RESEARCH ARTICLE

Static and dynamic quantile preferences

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Abstract

This paper axiomatizes static and dynamic quantile preferences. Static quantile preferences specify that a prospect should be preferred if it has a higher τ-quantile, for some $\tau \in (0, 1)$, while its dynamic counterpart extends this to take into account a sequence of decisions and information disclosure. An important motivation for the axiomatization that leads to this preference is the separation of tastes and beliefs. We first axiomatize quantile preferences for the static case with finite state space and then extend the axioms to the dynamic context. The dynamic preferences induce an additively separable quantile model with standard discounting, that is, the recursive equation is characterized by the sum of the current period utility function and the discounted value of the certainty equivalent, which is a quantile function. These preferences are time consistent and have a simple quantile recursive representation, which gives the model the analytical tractability needed in several fields in financial and economic applications. Finally, we study the notion of risk attitude in both the static and recursive quantile models. In quantile models, the risk attitude is completely captured by the quantile τ, a single-dimensional parameter. This is simpler than in expected utility models, where in general the risk attitude is determined by a function.

Keywords Quantile preferences · Recursive utility · Ordinality · Axioms

JEL Classification C60 · D81

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1 Introduction

Since Daniel Bernoulli's solution to the St. Petersburg's Paradox in 1738, expected utility (EU) has become the standard model for studying choices under risk. Nevertheless, this framework has been subjected to many criticisms and disqualifying evidence. As a result, in the intervening almost 300 years of research, and especially in the last three decades, many alternative models have been considered and thoroughly scrutinized. It is interesting, however, to observe how little attention has been devoted to the study of quantile utility maximization as an alternative model, even though quantile maximization is simple and, at least in some settings, provides a compelling choice. For instance, quantiles have been used for decision making in banking and investment (in the form of Value-at-Risk), in the mining, oil and gas industries (in the form of "probabilities of exceeding" a certain level of production), and importantly, quantile maximization offers a simple solution to the St. Petersburg's Paradox.[1]

In this paper, we are concerned with quantile preferences. Namely, a prospect (or random variable) X is considered better than Y if its τ-quantile is larger: $\mathrm{Q}_\tau[X] > \mathrm{Q}_\tau[Y]$, for a given $\tau \in (0, 1)$. Manski (1988) was the first to study such preferences, which were recently axiomatized by Chambers (2009) and Rostek (2010). Rostek (2010) axiomatized the quantile preferences in the context of Savage (1954)'s subjective uncertainty framework, while Chambers (2009) works in a risk setting, where the utility function and the probability distributions are already fixed. Rostek (2010) discusses several advantages of the static quantile preference, such as robustness, ability to deal with categorical (instead of continuous) variables, and the flexibility of offering a family of preferences indexed by quantiles. We suggest a new justification for quantile preferences: the separation of tastes and beliefs. The relevance of this objective was put forward by Ghirardato et al. (2005). Since "the tastes of a decision maker are quite stable, while his beliefs may change in the course of making decisions," they argue that it is "customary to seek representations of preferences which cleanly separate the tastes component from the beliefs component" (p. 129–130). They do not insist in a complete separation of tastes and beliefs as we do here, however, because this complete separation would rule out many preferences usually considered by decision theorists (see their Remark 1, p. 133 and the discussion in Sect. 3 below). The separation of tastes and beliefs leads to simplicity, objectivity, and transparency in choices.[2]

Applications of quantile preferences have been recently studied. For instance, Bhattacharya (2009) studies the problem of optimally dividing individuals into peer groups to maximize social gains from heterogeneous peer effects using a quantile utility distribution for the planner. de Castro et al. (2019) develop a model for optimal portfolio allocation for an investor with quantile preferences. From an experimental point of view, de Castro et al. (2020) find that the behavior of between 30% and 50% of the individuals can be better described with quantile preferences rather than EU.

[1] Indeed, recall that the problem in the St. Petersburg's Paradox is the fact that the presented lottery has an infinite expectation and, therefore, the received wisdom at the time—maximize expected values—was not applicable. In contrast, this lottery (as any other) has a finite τ-quantile for any $\tau \in (0, 1)$.

[2] Example 3.3 in Sect. 3.1 discusses an empirical setting where this separation is important.

Dynamic quantile preferences have also been studied by recent works. For instance, Giovannetti (2013) studies a two-period economy for an asset pricing model under quantile preferences maximization. de Castro and Galvao (2019) suggest a dynamic model of rational behavior under uncertainty, in which the agent maximizes the stream of the future τ-quantile utilities, but do not provide an explicit axiomatization for the recursive quantile preferences. This paper fulfills this gap. Although quantiles do not share some of the helpful properties of expectations, such as linearity and the law of iterated expectations, de Castro and Galvao (2019) establish all the standard results in dynamic models.[3] The dynamic quantile preferences have several advantages as they allow for the separation between risk aversion and elasticity of intertemporal substitution (EIS), which the standard EU model is not able to deliver, while maintaining monotonicity. Note that Epstein and Zin (1989) and Weil (1990) are the alternative framework mostly used to obtain the separation between EIS and risk aversion, in applied works, but since they do not satisfy monotonicity, they may lead to some unusual behavior.[4]

This paper contributes to the literature by providing axiomatizations of the static and the recursive quantile preferences. A central objective is to establish foundations for the dynamic preference, but for technical reasons, we are not able to use the previous existing axiomatizations for the static case and develop an alternative axiomatization of the static quantile preference. In particular, we are not able to appeal directly to the results of Chambers (2009) nor Rostek (2010) because the former works in a setting of risk, instead of uncertainty, while the latter requires an infinite state space, which is not convenient for the axiomatization of the dynamic preferences. We present an axiomatization of a preference over risky prospects based on the quantile of the distribution of outcomes. This choice rule is discussed, justified and deduced from axioms both in the static (with just one period) and dynamic (with many periods) settings.

The main axioms that provide the quantile preferences representation in the static case are Monotonicity, Ordinality, and Betting Consistency. Monotonicity stipulates that if for each state of nature, the consequence of some act f is preferred to that of another act g, then f is preferred to g and it is a very standard property, valid by most preferences. Ordinality requires that increasing transformations of a pair of acts do not change their ranking. Ordinality is central for the developments in this paper. Finally, Betting Consistency allows writing the aggregation operator as a quantile with respect to some probability measure.

Next we develop an axiomatization of the recursive quantile preferences. For this, we apply the results in Bommier et al. (2017). When extending the results to the quantile recursive case, in addition to the standard axioms in Bommier et al. (2017), we extend the Monotonicity, Ordinality, and Betting Consistency axioms from the static to the dynamic context. We show that the dynamic quantile preferences induce an additively separable quantile model with standard discounting, that is, the recursive

[3] de Castro and Galvao (2019) show that the quantile preferences are dynamically consistent, the corresponding dynamic problem yields a value function, via a fixed point argument, this value function is concave and differentiable, the principle of optimality holds, and derive the corresponding Euler equation.

[4] See Bommier et al. (2017) and its discussion of their axiom D7 and the Epstein–Zin–Weil preference (see Epstein and Zin 1989; Weil 1990). See also de Castro and Galvao (2019).

equation is characterized by the sum of the current period utility function and the discounted value of the certainty equivalent, which is a quantile function. The recursive equation is the central element in the definition of the dynamic quantile preferences. Thus, this paper complements de Castro and Galvao (2019) by providing a formal axiomatization for the dynamic model of rational behavior under uncertainty.

Finally, we examine the notion of risk attitude in both static and recursive quantile models. The notion of comparative risk attitude in quantile models is captured by the single dimensional quantile $\tau \in (0, 1)$. This is in contrast with the EU models, where the risk attitude needs to be defined by a function.

The remaining of the paper is organized as follows. Section 2 presents the setting and definitions. Section 3 develops an axiomatization for the static model. Section 4 provides an axiomatization for the recursive quantile preference. Section 5 discusses the risk attitude for static and dynamic quantile models. Finally, Sect. 6 concludes. We relegate all proofs to the Appendix.

2 Setting and definitions

This section defines quantiles and introduces terminology used later. Given a random variable (r.v.) X, let F_X (or simply F) denote its cumulative distribution function (c.d.f.), that is, $F_X(\alpha) \equiv \Pr[X \leq \alpha]$. The quantile function $Q : [0, 1] \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is the generalized inverse of F:

$$Q(\tau) \equiv \begin{cases} \inf\{\alpha \in \mathbb{R} : F(\alpha) \geq \tau\}, \text{ if } \tau \in (0, 1] \\ \sup\{\alpha \in \mathbb{R} : F(\alpha) = 0\}, \text{ if } \tau = 0. \end{cases}$$

The definition is special for $\tau = 0$ so that the quantile assumes a value in the support of X.[5] It is clear that if F is invertible, that is, if F is strictly increasing, its generalized inverse coincide with the inverse, that is, $Q(\tau) = F^{-1}(\tau)$. Usually, it will be important to highlight the random variable to which the quantile refers. In this case we will denote $Q(\tau)$ by $\mathrm{Q}_\tau[X]$ or by $\mathrm{Q}_\tau^p[X]$ if it is convenient to be explicit about the probability p on the underlying space. The cases $\tau = 0$ and $\tau = 1$ are not relevant for most applications and bring additional technical difficulties that would require us to state special cases in our results. For those reasons, we will follow de Castro and Galvao (2019) and restrict our attention to the case $\tau \in (0, 1)$.

A well-known and useful property of quantiles is "invariance" with respect to monotonic transformations, that is, if $\xi : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function, then[6]

$$\mathrm{Q}_\tau[\xi(X)] = \xi\left(\mathrm{Q}_\tau[X]\right). \tag{1}$$

See other properties of quantiles in de Castro and Galvao (2019).

[5] Indeed, $\inf\{\alpha \in \mathbb{R} : F(\alpha) \geq 0\} = -\infty$, no matter what is the distribution.

[6] In fact, (1) holds for (weakly) increasing and left-continuous $\xi : \mathbb{R} \to \mathbb{R}$. See de Castro and Galvao (2019, Lemma A.2, p. 1927).

Let S denote the finite set of states of the world and let Σ be an algebra of subsets of S. Any $E \in \Sigma$ is called an *event*. The topological space Υ is called the set of *consequences*. Any Σ-measurable function $f : S \to \Upsilon$ that takes only finitely many values is called an *act* and $\mathcal{F}$ denotes the set of all acts. Endow $\mathcal{F}$ with the product topology of Υ^S. An event E is *null* if for all $f, g \in \mathcal{F}$, if $(f(s) = g(s), \forall s \notin E)$ implies $f \sim g$. We denote by xEy the act $f \in \mathcal{F}$ defined by $f(s) = x$ if $s \in E$ and $f(s) = y$ if $s \notin E$. As usual, Υ is seen as a subset of $\mathcal{F}$. Also as usual, we write $f \succ g$ if $(f \succcurlyeq g) \wedge \neg(g \succcurlyeq f)$, $f \sim g$ if $(f \succcurlyeq g) \wedge (g \succcurlyeq f)$, $f \preccurlyeq g$ if $g \succcurlyeq f$ and analogously for $\prec$.

We denote by $B_0(\Sigma)$ the set of all real-valued Σ-measurable real-valued simple functions on S. Thus, if $u : \Upsilon \to \mathbb{R}$ and $f \in \mathcal{F}$, $u(f) = u \circ f \in B_0(\Sigma)$. We denote by $B(\Sigma)$ the vector space of the real-valued, bounded, Σ-measurable functions. The subset of $B_0(\Sigma)$ (respectively, of $B(\Sigma)$) comprised of the functions that take values in $K \subseteq \mathbb{R}$ is denoted by $B_0(\Sigma, K)$ (respectively, $B(\Sigma, K)$). If $E \in \Sigma$, we denote by $1_E \in B(\Sigma, [0, 1])$ the indicator function of $E \subset S$, that is, $1_E(s) = 1$ if $s \in E$ and $1_E(s) = 0$ otherwise. We will abuse notation and denote, for any $k \in \mathbb{R}$, $k1_S$ by simply k.

We will need to assume further structure for the set of consequences Υ or the set of acts $\mathcal{F}$ that will be discussed below.

3 Static quantile preferences

This section provides two alternative axiomatizations of the static quantile preferences. This is a building block for the axiomatization of the recursive preferences in the next section. Section 3.1 presents an axiomatization in the familiar Anscombe–Aumann setting. In this subsection we also discuss the separation of tastes and beliefs as studied by Ghirardato et al. (2005). Section 3.2 presents an axiomatization for the topological case. The axiomatization in this setting is important because it will be used later, in the axiomatization of the dynamic preferences. Both axiomatizations include Ordinality, to be defined below, as its main axiom.

3.1 Axiomatization in the Anscombe–Aumann setting

Here we assume that Υ is a convex subset of a vector space. This assumption is usually justified by thinking that Υ is the set of all the lotteries on a set of prizes, as in Anscombe and Aumann (1963). Given $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$, we denote by $\alpha f + (1 - \alpha)g$ the act that yields $\alpha f(s) + (1 - \alpha)g(s)$ if $s \in S$ obtains. Let $K \subseteq \mathbb{R}$ and $0 \in int(K)$. A functional $I : B_0(\Sigma, K) \to \mathbb{R}$ is a *certainty equivalent* if and only if, for all $\varphi, \psi \in B_0(\Sigma, K)$, $k \in K$ and $\alpha \in (0, 1)$ the following conditions are satisfied:[7]

(i) $\varphi(s) \geq \psi(s) \forall s \in S$ implies $I[\varphi] \geq I[\psi]$;

[7] This corresponds to normalized niveloids, in the terminology of Maccheroni et al. (2006a); see in particular their Lemma 25. We adopt the terminology that is more usual in the decision theory literature. See, e.g., Bommier et al. (2017) and Strzalecki (2013) for more discussion.

(ii) $I[\alpha\varphi + (1-\alpha)k] = I[\alpha\varphi] + (1-\alpha)k$;
(iii) $I[k] = k$.

The following axioms about the binary relation $\succcurlyeq$ on $\mathcal{F}$ are standard and taken from Maccheroni et al. (2006a). See the paper for a detailed discussion.

Axiom Q1 (*Monotonic, Non-trivial Weak order*). The binary relation $\succcurlyeq$ is complete, transitive, non-trivial (that is, there exist $f, g \in \mathcal{F}$ such that $f \succ g$) and monotonic (that is, for any $f, g \in \mathcal{F}$, $(f(s) \succcurlyeq g(s), \forall s \in S) \Rightarrow f \succcurlyeq g$).
Axiom Q2l (*Continuity*). If $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succcurlyeq h\}$ and $\{\alpha \in [0,1] : h \succcurlyeq \alpha f + (1-\alpha)g\}$ are closed.
Axiom Q3 (*Weak Certainty Independence*). If $f, g \in \mathcal{F}$, $x, y \in \Upsilon$, and $\alpha \in (0,1)$, $\alpha f + (1-\alpha)x \succcurlyeq \alpha g + (1-\alpha)x \Rightarrow \alpha f + (1-\alpha)y \succcurlyeq \alpha g + (1-\alpha)y$.

These axioms lead to a general and useful representation:

Lemma 3.1 (Maccheroni et al. 2006a, Lemma 28, p. 1477) *A binary relation $\succcurlyeq$ on $\mathcal{F}$ satisfies Axioms Q1–Q3 if and only if there exist a nonconstant affine function $u : \Upsilon \to \mathbb{R}$ and a certainty equivalent $I_u : B_0(\Sigma, u(\Upsilon)) \to \mathbb{R}$ such that*

$$f \succcurlyeq g \iff I_u[u \circ f] \geq I_u[u \circ g]. \tag{2}$$

The above representation suggests the following interpretation of the separation of "tastes" and "beliefs". While $u : \Upsilon \to \mathbb{R}$ captures the "tastes" of the decision maker (DM) over consequences, the certainty equivalent I_u captures the DM's "beliefs." This interpretation is mentioned by Strzalecki (2013, p. 1044) and Ghirardato et al. (2005), among others. This analysis is based on two observations. First, the utility $u : \Upsilon \to \mathbb{R}$ indeed captures the preference over consequences, that is, for any $x, y \in \Upsilon$,

$$x \succcurlyeq y \iff u(x) \geq u(y). \tag{3}$$

Thus, the "tastes over consequences" are completely defined by $u : \Upsilon \to \mathbb{R}$. Second, the uncertainty over the states $s \in S$, on the other hand, is relevant and thus captured only by $I_u : B_0(\Sigma, u(\Upsilon)) \to \mathbb{R}$.[8] For example, if Q3 is strengthened to the usual Independence axiom, we obtain the expected utility representation and I_u is an expectation operator that is pinned down by the DM's beliefs over the states $s \in S$.

As mentioned in the introduction above, Ghirardato et al. (2005) argue that it is desirable that the representation in (2) leads to a separation of "tastes" and "beliefs". By this separation, they mean the following: suppose that $v : \Upsilon \to \mathbb{R}$ is another utility function that represents the preference over the consequences Υ:

$$x \succcurlyeq y \iff v(x) \geq v(y). \tag{4}$$

[8] Notice that I_u can capture a part of the risk attitude or uncertainty attitude, that is, some "taste" over risk or uncertainty, as Remark 3.6 below discuss in more detail. This is why we are emphasizing "tastes over consequences" when talking about u.

Presumably, we could also have a certainty equivalent $I_v : B_0(\Sigma, u(\Upsilon)) \to \mathbb{R}$ such that

$$f \succcurlyeq g \iff I_v[v \circ f] \geq I_v[v \circ g]. \tag{5}$$

Notice that the only difference between (2) and (5) is the change in the utility function capturing the tastes over consequences. Since the beliefs do not change, we should have $I_u = I_v$. This equality is exactly what they mean by complete separation of tastes and beliefs. More formally,

Definition 3.2 A binary relation $\succcurlyeq$ on $\mathcal{F}$ for which (2) and (5) hold satisfy:

(i) complete separation of beliefs and tastes (over consequences) if (3) and (4) imply $I_u = I_v$.
(ii) partial (or cardinal) separation of beliefs and tastes (over consequences) if (3), (4) and the fact that v is a cardinal transformation of u (that is, there exists $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $v(x) = \alpha u(x) + \beta$) imply $I_u = I_v$.

Thus, we have complete separation of tastes and beliefs if the functional does not depend on the utility over consequences. The separation is partial or cardinal if the functional is fixed only for cardinal transformations of the utility. Ghirardato et al. (2005) focus only on partial separation of tastes and beliefs, while we focus on complete separation.

The separation of tastes and beliefs is desirable and important in some, *but certainly not all*, circumstances. An example where we believe the complete separation is important is given by the following:[9]

Example 3.3 Assume that the DM works in a national development bank and evaluates applications for financing public projects in a given country. The projects will lead to outcomes in a set of consequences Υ, but are subjected to uncertain conditions, modeled as a set S. Thus, projects may be viewed as functions $f : S \to \Upsilon$. Using data from past similar projects, she is able to construct a model that derives beliefs over the states $s \in S$. The DM needs to explain her choices of projects to the board of the public development bank.

The members of the board, which may represent different segments of the country population, may have different attitudes towards risk. They all agree that more money is more valuable than less, but some are more risk averse than others. Therefore, the utility functions $u : \Upsilon \to \mathbb{R}$ and $v : \Upsilon \to \mathbb{R}$ of two board members may agree on the order over consequences (that is, (3) and (4) hold), but v is not a cardinal transformation of u.

The data from past similar projects helps to pin down beliefs and define a functional $I : B_0(\Sigma) \to \mathbb{R}$ to support her decision over projects. We assume that this functional was previously agreed upon, based on the data. Her criterion has the following property: project $f : S \to \Upsilon$ is at least as good as project $g : S \to \Upsilon$ if and

[9] The example is inspired in a real-world case. The authors have learned that the Brazilian's Development Bank (BNDES), when evaluating the prospects of projects on energy, use a quantile criterion to determine whether or not the project should be financed.

only if

$$I[u \circ f] \geq I[u \circ g] \iff I[v \circ f] \geq I[v \circ g],$$

for any functions $u, v : \Upsilon \to \mathbb{R}$ such that $\forall x, y \in \Upsilon$, $u(x) \geq u(y) \iff v(x) \geq v(y)$. In this way, the DM may argue that her criterion does not favor any particular member of the board and treats them equally. She can do this because her criterion allows a complete separation of tastes and beliefs. Actually, if the projects' consequences are given already in real values (money), then her criterion can be summarized simply as

$$f \succcurlyeq g \iff I[f] \geq I[g].$$

Thus, the DM does not need to use any arbitrary function $u : \Upsilon \to \mathbb{R}$. Her criterion is completely determined by the functional I. This leads to a transparent procedure to rank projects, reducing the possibility of manipulation and corruption in the choices of the public development bank.

Remark 3.4 In this example, the objectivity of the procedure, with minimal subjective elements, is the central quality. Since objectivity and subjectivity are the main themes of Gilboa et al. (2010), it is useful to observe that our treatment of those concepts is slightly different from theirs. They say that a choice "is *objectively rational* if the DM can convince others that she is right in making it," while it "is *subjectively rational* if others cannot convince the DM that she is wrong in making it." The formalization of those notions lead them to an *objective* preference that is incomplete and a *subjective* preference that completes it. Their objective preference is incomplete because $X \succcurlyeq Y$ only if $\mathrm{E}_P[u(X)] \geq \mathrm{E}_P[u(Y)]$ for *all* probability measures P in a set of possible probabilities $\mathcal{P}$. Since the inequality can be reversed for some pair of acts and distinct probabilities in $\mathcal{P}$, this criterion cannot rank all pairs. The same set of probabilities $\mathcal{P}$ appears in the subjective preference, which is a Gilboa and Schmeidler (1989)'s Maximin Expected Utility, and displays ambiguity aversion. On the other hand, ambiguity aversion does not play a role in Example 3.3. The subjectivity that Example 3.3 rules out is in the choice of the utility function u. The functional I is considered objective, because it "was previously agreed upon, based on the data." The reader should be warned that subjectivity is not completely ruled out however, since the functional I itself may contain subjective elements, as we further discuss in Remark 3.6 below.

Having justified the complete separation of tastes and beliefs as a criterion desirable in *some* settings, we turn to translate this property into an axiom. First notice that (3) and (4) hold in general if there exists a strictly increasing function $\xi : \mathbb{R} \to \mathbb{R}$ such that $v = \xi \circ u$. Therefore, we have complete separation of tastes and believes if the certainty equivalent $I : B_0(\Sigma) \to \mathbb{R}$ satisfies

$$I[\xi \circ u \circ f] = \xi\left(I[u \circ f]\right), \tag{6}$$

for any $\xi : \mathbb{R} \to \mathbb{R}$ that is strictly increasing, and any $u : \Upsilon \to \mathbb{R}$ and $f \in \mathcal{F}$. Restricting ξ to be strictly increasing and continuous and considering a larger domain, Chambers (2007, 2009) names property (6) Ordinal Covariance. More precisely, a functional $I : B(\Sigma) \to \mathbb{R}$ satisfies Ordinal Covariance if

$$I[\xi \circ h] = \xi \left(I[h] \right), \forall h \in B(\Sigma). \tag{7}$$

We cannot use (6) nor (7) because they are properties of the function I, not properties of the preference $\succcurlyeq$.[10] Instead, we introduce a new axiom, that we call "Ordinality". To state it, we need the following definition:

Definition 3.5 A function $\varphi : \Upsilon \to \Upsilon$ is *increasing* if $x \succcurlyeq y \Rightarrow \varphi(x) \succcurlyeq \varphi(y), \forall x, y \in \Upsilon$ and it is *strictly increasing* if it is increasing and $x \succ y \Rightarrow \varphi(x) \succ \varphi(y), \forall x, y \in \Upsilon$.

We can now state our main axiom.

Axiom Q4 (*Ordinality*). For any increasing $\varphi : \Upsilon \to \Upsilon$, we have

$$f \succcurlyeq g \Rightarrow \varphi(f) \succcurlyeq \varphi(g). \tag{8}$$

Ordinality is the central axiom of our axiomatization of static quantile preferences. It requires that the preference is ordinal, that is, the only importance for the ranking of any act is the order of the consequences, not its cardinal value. Ordinality is a variation of Ordinal Invariance axiom presented in Chambers (2005, section 9).[11] For Chambers (2005), $\mathcal{F}$ is the set of real-valued, bounded, Σ-measurable functions, that is, $B(\Sigma)$ in our notation, and he says that a preference $\succcurlyeq$ on this $B(\Sigma)$ satisfies *Ordinal Invariance* if for all $f, g \in B(\Sigma)$ and all strictly increasing and continuous $\varphi : \mathbb{R} \to \mathbb{R}$,

$$f \succcurlyeq g \Leftrightarrow \varphi(f) \succcurlyeq \varphi(g). \tag{9}$$

There are a number of differences between Ordinal Invariance and Ordinality. First, for Ordinality, the acts in $\mathcal{F}$ may take values in a space of consequences Υ, which is not necessarily the reals. The order on Υ is inherited from the other on $\mathcal{F}$, since Υ may be viewed as subset of $\mathcal{F}$ (as usual, the elements of Υ can be identified with constant acts).

Second, Ordinal Invariance applies to strictly increasing and continuous $\varphi : \mathbb{R} \to \mathbb{R}$, while Ordinality applies to increasing $\varphi : \Upsilon \to \Upsilon$. The fact that Ordinality dispenses with continuity of φ makes even the implication $\Rightarrow$ not trivial. In fact, it is not true in general for any continuous model, as discussed in detail in Sect. 3.3 below. Since φ is not restricted to be strictly increasing, the implication is required in just one direction. Indeed, the reverse direction would be false for increasing functions. To see this, consider a constant function $\varphi : \Upsilon \to \Upsilon$, which is increasing and makes the right hand side of (8) hold with indifference, that is, $\varphi(f) \sim \varphi(g)$. In this case the reverse implication $\Leftarrow$ would be false if $g \succ f$. In fact, the equivalence cannot be imposed even if we require φ to be strictly increasing. If φ is strictly increasing *with inverse* φ^{-1},

10 Chambers (2005, 2007) defined related properties directly on preferences. See discussion below, after the introduction of Ordinality.

11 See also Chambers (2007, footnote 2, p. 422) for a related condition.

then we could apply Q4 to φ^{-1} (easily seen to be also strictly increasing), to obtain $f \succcurlyeq g \Leftrightarrow \varphi(f) \succcurlyeq \varphi(g)$.[12] The problem is that a strictly increasing map $\varphi : \Upsilon \to \Upsilon$ may fail to be invertible for a general space Υ, because of the indifferences. For instance, consider $\Upsilon = \{a, b, c\}$ with $a \sim b \succ c$ and $\varphi(a) = \varphi(b) = a$ and $\varphi(c) = c$. This φ is strictly increasing but not invertible. In sum, by considering increasing maps and working with the implication in just one direction, Ordinality circumvents all these difficulties in a succinct form.

It is well known that we need an extra axiom to deal with probabilities in finite sets, as we are considering here. We follow Chambers (2007) in naming it Betting Consistency, but it appears in a variety of previous papers dealing with finite sets; see, for instance, Fishburn (1986, Axiom 4, p. 229).

Axiom Q5 (*Betting Consistency*). Let $\{A_1, \ldots, A_n\} \subset \Sigma$ and $\{B_1, \ldots, B_n\} \subset \Sigma$ be such that $\sum_{i=1}^{n} 1_{A_i} \geq \sum_{i=1}^{n} 1_{B_i}$. Then, there exists $i \in \{1, \ldots, m\}$ such that $\overline{x}\ A_i\ \underline{x} \succcurlyeq \overline{x}\ B_i\ \underline{x}$, for any $\overline{x}, \underline{x} \in \Upsilon$ such that $\overline{x} \succ \underline{x}$.

We can now state our first main result.

Theorem 1 *A binary relation $\succcurlyeq$ on $\mathcal{F}$ satisfies Axioms Q1–Q5 if and only if there exist a nonconstant affine function $u : \Upsilon \to \mathbb{R}$, probability $p : \Sigma \to [0, 1]$ and $\tau \in (0, 1)$ such that*

$$f \succcurlyeq g \iff I[u \circ f] \geq I[u \circ g],$$

where $I : B_0(\Sigma, u(\Upsilon)) \to \mathbb{R}$ is given by

$$I[\xi] = \inf\{\alpha : p(\{s \in S : \xi(s) \geq \alpha\}) \leq \tau\} = \mathrm{Q}_\tau^p[\xi]$$

for any $\xi \in B_0(\Sigma, u(\Upsilon))$.

Remark 3.6 As we discuss in Sect. 5, the risk attitude in quantile models is captured by τ. Therefore, it is associated to the functional $I : B_0(\Sigma, u(\Upsilon)) \to \mathbb{R}$ and not to the utility function $u : \Upsilon \to \mathbb{R}$. This is why we have emphasized that the separation is of beliefs and "tastes over consequences." This also clarifies a comment made at the end of Remark 3.4, where we warned that the functional I may contain subjective elements: it includes the parameter τ, associated to the risk attitude. In other words, the functional includes "taste for risk," but not "taste over consequences."

3.2 Axiomatization in the topological setting

We now present an axiomatization of quantile preferences in a topological setting. This is important for us because it will be used later, in the derivation of the dynamic quantile preferences. Thus, in this subsection, we assume that Υ (and therefore, $\mathcal{F}$) is a connected separable topological space. Recall that $\mathcal{F}$ is endowed with the product

[12] This equivalence holds for a quantile preference because we focus on simple acts. In the general case, it is necessary to require extra conditions, as left-continuity; see de Castro and Galvao (2019).

topology of Υ^S. The role of this assumption is to guarantee the existence of a continuous utility function representing the (continuous) preference $\succcurlyeq$; see Debreu (1964). In this setting, we modify the continuity assumption as follows:

Axiom Q2' (*Continuity*). For any $f \in \mathcal{F}$, the sets $\{g \in \mathcal{F} : g \succcurlyeq f\}$ and $\{g \in \mathcal{F} : f \succcurlyeq g\}$ are closed.

In this setting, we do not need Q3 (Weak Certainty Independence), while Ordinality and Betting Consistency remain the same. We have:

Theorem 2 *The preference $\succcurlyeq$ satisfies axioms Q1, Q2', Q4 and Q5 if and only if there exists a nonconstant continuous utility function $u : \Upsilon \to [0, 1]$, a probability $p : \Sigma \to [0, 1]$ and $\tau \in (0, 1)$ such that $U : \mathcal{F} \to [0, 1]$ represents $\succcurlyeq$, where*

$$U(f) \equiv \inf\{\alpha : p(\{s \in S : u(f(s)) \geq \alpha\}) \leq \tau\}. \tag{10}$$

The result in Theorem 2 is a building block used in the next section to establish the axiomatization of the recursive quantile preferences. Notice that the "uniqueness" of u is asserted only with respect to monotonic transformations, and not only *affine* transformations as in the expected utility case. On the other hand, the representation is equivalent for any strictly increasing transformations of u.

Remark 3.7 The methods proposed in this paper are based on quantiles, which are deeply related to Value-at-Risk (VaR)—formally, they are the same object—and VaR is a measure of risk in Artzner et al. (1999)'s terminology (although not a coherent one). However, the objectives and concepts related to quantile preferences are different from those of (coherent) risk measures. A purpose of risk measures is to restrict the set of acceptable choices by a portfolio manager, but it is not intended to guide her in the specific portfolio selection. VaRs are also commonly used to restrict the set of acceptable portfolios and the τ's usually considered are either 0.05 or 0.01, but our proposal is to axiomatize the quantile preferences, which could be used to choose the portfolio directly.

3.3 Ordinality and quantiles

The purpose of this section is to investigate when the axiom of Ordinality Q4 holds for quantiles in different settings. The necessity part of Theorems 1 and 2 establishes that Ordinality (Q4) holds for a quantile preference in $\mathcal{F}$. In order to clarify what we mean by the validity of Ordinality for "different settings," let us say that a preference $\succcurlyeq$ on a subset $\mathcal{F}$ of the set of Σ-measurable functions $f : S \to \Upsilon$ is a quantile preference if the following holds:

$$f \succcurlyeq g \iff \mathrm{Q}_\tau^p[u(f)] \geq \mathrm{Q}_\tau^p[u(g)], \tag{11}$$

where $u : \Upsilon \to \mathbb{R}$, $\mathrm{Q}_\tau^p[u(f)] = \inf\{\alpha : p(\{s \in S : u(f(s)) \geq \alpha\}) \leq \tau\}$ and (S, Σ, p) is a probability space. Given (11), a quantile preference satisfies Ordinality if and only if for all $f, g \in \mathcal{F}$ and an increasing $\varphi : \Upsilon \to \Upsilon$, the following holds:

$$\mathrm{Q}_\tau^p[u(f)] \geq \mathrm{Q}_\tau^p[u(g)] \implies \mathrm{Q}_\tau^p[u(\varphi(f))] \geq \mathrm{Q}_\tau^p[u(\varphi(g))]. \tag{12}$$

Interestingly, Ordinality (12) does not hold in all cases, but it does hold in the most important cases, namely, when the random variables assume only finitely many values or when their induced distribution is atomless. Example 3.11 below contains a counterexample for the general case, where we have both a continuous variable and a discrete one.

To explain those results, we begin with the setting of simple acts in which Theorems 1 and 2 are stated. As we mentioned above, the necessity part of those theorems establish that Ordinality holds in this setting. Because of its importance for our discussion, we state this result separately here:

Proposition 3.8 *Let (S, Σ, p) be a probability space and assume that $f, g : S \to \Upsilon$ are measurable functions assuming a finite number of values. Then, (12) holds.*

Proposition 3.9 below shows that Ordinality also holds if the variables are atomless. We say that $f : S \to \Upsilon$ is atomless (or has no atom) if $p(\{s \in S : f(s) = a\}) = 0$ for all $a \in \Upsilon$. We have the following:

Proposition 3.9 *Let (S, Σ, p) be a probability space and assume that $f, g : S \to \Upsilon$ are measurable functions with no atoms. Then, (12) holds.*

We will actually prove a slightly stronger result, that completes the characterization of when (12) holds. For simplicity of notation, we will write $\mathrm{Q}_\tau^p[\cdot]$ as $\mathrm{Q}_\tau[\cdot]$.

Proposition 3.10 *Assume that*

$$p(\{s \in S : u(f(s)) < \mathrm{Q}_\tau[u(f)]\}) < \tau \text{ or } p(\{s \in S : u(g(s)) < \mathrm{Q}_\tau[u(g)]\}) \geq \tau. \tag{13}$$

Then, (12) holds.

In other words, (12) can fail only if

$$p(\{s \in S : u(f(s)) < \mathrm{Q}_\tau[u(f)]\}) \geq \tau > p(\{s \in S : u(g(s)) < \mathrm{Q}_\tau[u(g)]\}).$$

Example 3.11 below shows that assumption in Eq. (13) cannot be dispensed with, that is, (12) may fail if the assumption is false.

Example 3.11 Let $S = [0, 1]$, Σ the usual Borel σ-algebra and p the Lebesgue measure on S. Let $u(x) = x$, $\Upsilon = [0, 1]$ and consider the following functions:

$$f(s) = s, \forall s \in S;$$

$$g(s) = \begin{cases} 0.25, & \text{if } s \in [0, 0.25) \\ 0.5, & \text{if } s \in [0.25, 0.5) \\ 0.75, & \text{if } s \in [0.5, 0.75) \\ 1, & \text{if } s \in [0.75, 1] \end{cases}; \quad \text{and } \varphi(x) = \begin{cases} x, & \text{if } x \in [0, 0.5) \\ 1, & \text{if } x \in [0.5, 1] \end{cases}.$$

Therefore, f is atomless and g has atoms in $\{0.25, 0.5, 0.75, 1\}$, each with 0.25 probability. Moreover, φ is increasing. Let $\tau = 0.5$. Then it is easy to see that $\mathrm{Q}_\tau[u(f)] = \mathrm{Q}_\tau[u(g)] = 0.5$. On the other hand,

$$\varphi(f(s)) = \begin{cases} s, & \text{if } s \in [0, 0.5) \\ 1, & \text{if } s \in [0.5, 1] \end{cases}; \qquad \varphi(g(s)) = \begin{cases} 0.25 & \text{if } s \in [0, 0.25) \\ 1 & \text{if } s \in [0.25, 1] \end{cases}.$$

With this, we obtain

$$\mathrm{Q}_\tau[u(\varphi(f))] = \inf\{\alpha : p(\{s \in S : u(\varphi(f(s))) \geq \alpha\}) \leq 0.5\} = 0.5;$$
$$\mathrm{Q}_\tau[u(\varphi(g))] = \inf\{\alpha : p(\{s \in S : u(\varphi(g(s))) \geq \alpha\}) \leq 0.5\} = 1.$$

That is, (12) fails: $\mathrm{Q}_\tau[u(f)] = \mathrm{Q}_\tau[u(g)] = 0.5$ and $\mathrm{Q}_\tau[u(\varphi(f))] = 0.5 < 1 = \mathrm{Q}_\tau[u(\varphi(g))]$. Notice also that this example fails condition (13):

$$p\left(\{s \in S : u(f(s)) < \mathrm{Q}_\tau[u(f)] = 0.5\}\right) = 0.5 = \tau$$
$$\text{and } p\left(\{s \in S : u(g(s)) < \mathrm{Q}_\tau[u(g)] = 0.5\}\right) = 0.25 < 0.5 = \tau.$$

Once Proposition 3.10 is established (see the proof in the Appendix), Proposition 3.9 easily follows.

Proof of Proposition 3.9 Since f and g are assumed to be atomless, then

$$p\left(\{s \in S : u(g(s)) < \mathrm{Q}_\tau[u(g)]\}\right) = p\left(\{s \in S : u(g(s)) \leq \mathrm{Q}_\tau[u(g)]\}\right) \geq \tau, \quad (14)$$

where the last inequality is an known property of quantiles; see, for instance, de Castro and Galvao (2019, Lemma A.1(iv), p. 1926). Therefore, the assumption in Proposition 3.10 is satisfied and the conclusion follows from it. □

Remark 3.12 Under the representation (11), Ordinality is equivalent to (12). Thus, the above results show that Ordinality may fail for quantiles only in a very special case: when f is atomless and g has atoms precisely at the probability level τ. Indeed, if g has no atoms at the probability level τ, (14) above shows that the conditions of Proposition 3.10 are satisfied and (12) holds. On the other hand, if f and g are simple, Proposition 3.8 also establishes (12). Therefore, the only case left is, as we said, when f is atomless and g has atoms precisely at the probability level τ. Since in applications it is not usual to consider atomless acts together with simple ones, this limitation is not likely to cause problems.

4 Recursive quantile preferences

Now we provide an axiomatization of the recursive quantile preferences. To do so, we will apply the results in Bommier et al. (2017). We restrict the analysis to the particular case of stationary IID Markov.[13] The extension of the results to a general

[13] The notion of IID was first introduced in Epstein and Schneider (2003a) in the context of ambiguity for the case of max-min expected utility representation.

Markov case is left for future work. The notion of stationary IID relies on the following: (i) restriction of the analysis to cases where the passing of time has no impact on the structure of the domain of choice; and (ii) introduction of a set of assumptions implying that a decision maker who uses, at all dates, the same history independent preference relation is time-consistent. The setup and notation are taken verbatim from Bommier et al. (2017).

4.1 Setup and notation

For the axiomatization of the dynamic preferences, we follow the definitions and assumptions of Bommier et al. (2017). Let S be a finite set representing the states of the world to be realized in each period.[14] Assume that S has at least three elements and denote by $\Sigma := 2^S$ the associated algebra of events. The full state space is $\Omega \equiv S^\infty$, with a state $\omega \in \Omega$ specifying a complete history $(s_1, s_2, \ldots)$. In each period $t > 0$, the individual knows the partial history $(s_1, \ldots, s_t)$. Such knowledge can be represented by a filtration $\mathcal{G} = (\mathcal{G}_t)_t$ on Ω where $\mathcal{G}_0 := \{\emptyset, \Omega\}$ and, for every $t > 0$, $\mathcal{G}_t := \Sigma^t \times \{\emptyset, S\}^\infty$. Let $C = [\underline{c}, \bar{c}] \subset \mathbb{R}_{++}$ be the set of all possible consumption levels. A consumption plan, or an act, is a C-valued, $\mathcal{G}$-adapted stochastic process, that is, a sequence $h = (h_0, h_1, \ldots)$ such that $h_t : \Omega \to C$ is $\mathcal{G}_t$-measurable for every t. The set of all consumption plans is denoted by $\mathcal{H}$ and endowed with the topology of pointwise convergence.

We consider a binary relation $\succcurlyeq$ on $\mathcal{H}$ and introduce a set of axioms. Some notation is required first. Given an act $h \in \mathcal{H}$ and state $\omega \in \Omega$, let $h(\omega) \in C^\infty$ be the deterministic consumption stream induced by h in state $\omega \in \Omega$, that is, $h(\omega) = (h_0, h_1(\omega), \ldots)$. Moreover, for any act $h \in \mathcal{H}$ and any $s \in S$, we define the conditional act $h^s \in \mathcal{H}$ by

$$\forall \omega = (s_1, s_2, \ldots) \in \Omega : h^s(s_1, s_2, \ldots) = h(s, s_2, \ldots) = (h_0, h_1(s, s_2, \ldots), \ldots).$$

The act h^s is obtained from h when knowing that the first component of the state of the world is equal to $s \in S$. Notice that $h^s(s_1, s_2, \ldots)$ is independent of s_1. Given $h = (h_0, h_1, h_2, \ldots) \in \mathcal{H}$, let h^1 be defined by

$$\forall \omega = (s_1, s_2, \ldots) \in \Omega : h^1(s_1, s_2, \ldots) = (h_1(s_1, s_2, \ldots), h_2(s_1, s_2, \ldots) \ldots).$$

The set of those h^1 can be identified with $\mathcal{H}^S$. We can construct the continuation act $h^{s,1} \in \mathcal{H}$ from the conditional act h^s by removing the first-period consumption. Formally, for any act $h = (h_0, h_1, h_2, \ldots) \in \mathcal{H}$ and any $s \in S$, the continuation act $h^{s,1}$ is given by

$$\forall \omega = (s_1, s_2, \ldots) \in \Omega : h^{s,1}(s_1, s_2, \ldots) = (h_1(s, s_2, \ldots), h_2(s, s_2, \ldots), \ldots). \tag{15}$$

[14] A finite space S is also used in Maccheroni et al. (2006b).

The continuation act $h^{s,1}$ can be viewed as the consumption plan implied by h starting at date 1 (ignoring consumption at date $t = 0$) and where the information revealed at the beginning of date 1 (i.e., s_1) is equal to s.

Last, for any $c \in C$ and $h \in \mathcal{H}$, we define the concatenated act $(c, h) \in \mathcal{H}$ by

$$(c, h) : \omega = (s_1, s_2, \dots) \in \Omega \mapsto (c, h)(\omega) = (c, h(s_2, \dots)) \in C^\infty.$$

The notions of conditional, continuation, and concatenated acts are related to each other. In particular, the conditional act is the concatenation of first-period consumption and the continuation act. Formally, for $h = (h_0, h_1, \dots) \in \mathcal{H}$ and $s \in S$, we have $h^s = (h_0, h^{s,1})$. Moreover, any concatenated act (c, h) has continuation h. In mathematical terms, for any $c \in C$, $h \in \mathcal{H}$, and $s \in S$, we have $(c, h)^{s,1} = h$.

4.2 Axioms

We borrow the axioms below from Bommier et al. (2017, Appendix A). They are also related to those in Chew and Epstein (1990).

Axiom D1 (*Weak Order*).
The binary relation $\succcurlyeq$ is complete and transitive on $\mathcal{H}$.
Axiom D2 (*Continuity*).
For all $h \in \mathcal{H}$, the sets $\{\hat{h} \in \mathcal{H} \mid \hat{h} \succcurlyeq h\}$ and $\{\hat{h} \in \mathcal{H} \mid h \succcurlyeq \hat{h}\}$ are closed in $\mathcal{H}$.
Axiom D3 (*Recursivity*). For all acts $h = (h_0, h_1, \dots)$ and $\hat{h} = (\hat{h}_0, \hat{h}_1, \dots)$ in $\mathcal{H}$ with $h_0 = \hat{h}_0$,

$$\left(\forall s \in S, h^s \succcurlyeq \hat{h}^s\right) \implies h \succcurlyeq \hat{h}.$$

If, in addition, one of the former rankings is strict, then the latter ranking is strict.
Axiom D4 (*History Independence*).
For all acts $h = (h_0, h_1, \dots)$ and $\hat{h} = (h_0, \hat{h}_1, \dots)$ in $\mathcal{H}$, and $\hat{h}_0 \in C$,

$$(h_0, h_1, \dots) \succcurlyeq (h_0, \hat{h}_1, \dots) \iff (\hat{h}_0, h_1, \dots) \succcurlyeq (\hat{h}_0, \hat{h}_1, \dots).$$

Axiom D5 (*Stationarity*). For all $c \in C$ and $h, \hat{h} \in \mathcal{H}$, we have $(c, h) \succcurlyeq (c, \hat{h}) \iff h \succcurlyeq \hat{h}$.
Axiom D6 (*Monotonicity for Deterministic Prospects*). For all $c^\infty, \hat{c}^\infty \in C^\infty$, if $c^\infty \geq \hat{c}^\infty$, then $c^\infty \succcurlyeq \hat{c}^\infty$. The latter ranking is strict whenever $c^\infty \gneqq \hat{c}^\infty$.
Axiom D7 (*Monotonicity*).
For any h and $\hat{h}$ in $\mathcal{H}$,

$$\left(h(\omega) \succcurlyeq \hat{h}(\omega) \text{ for all } \omega \in \Omega\right) \implies h \succcurlyeq \hat{h}.$$

As discussed in Bommier et al. (2017), axioms D1 and D2 are standard. Axiom D3, recursivity, ensures that ex-ante choices remain optimal when they are evaluated ex-post. This axiom has appeared in the literature, see, e.g., Chew and Epstein

(1990). Axioms D4 and D5, history independence and stationarity, respectively, are complementary assumptions. These axioms are related to argument of Koopmans (1960) that "the passage of time does not have an effect on preferences." Axiom D6, monotonicity for deterministic prospects, requires that, in the absence of uncertainty, higher consumption is always better. Finally, axiom D7, monotonicity, is a consistency requirement between preferences over temporal lotteries and preferences over deterministic consumption streams. D7 stipulates that a temporal lottery is preferred whenever it provides a better consumption stream in every state of the world. Monotonicity also appears in Epstein and Schneider (2003b), Maccheroni et al. (2006b), and Kochov (2015).

4.3 Characterization of monotone recursive preferences

A function $I : B_0(\Sigma) \to \mathbb{R}_+$ is a *certainty equivalent* if it is continuous, strictly increasing and satisfies $I(x) = x$ for any $x \in \mathbb{R}_+$, where we see real numbers as constant functions in $B_0(\Sigma)$. A certainty equivalent $I : B_0(\Sigma) \to \mathbb{R}_+$ is *translation-invariant* if $I(x + f) = x + I(f)$ for any $x \in \mathbb{R}_+$ and $f \in B_0(\Sigma)$ and it is *scale-invariant* if for all $\lambda \in \mathbb{R}_+$ and $f \in B_0(\Sigma)$, $I(\lambda f) = \lambda I(f)$. Given a function $V : \mathcal{H} \to \mathbb{R}$ and an act $h \in \mathcal{H}$, we let $V \circ h^1$ denote the function $s \mapsto V(h^{s,1})$. If V is a utility function, then $V \circ h^1$ is the state-contingent profile of continuation utilities induced by the act h in period 1. A *time aggregator* is a function $W : C \times [0, 1] \to [0, 1]$.

We say that $\succcurlyeq$ admits a *recursive representation* (V, W, I) if

$$h \succcurlyeq h' \iff V(h) \succcurlyeq V(h'), \tag{16}$$

where $V : \mathcal{H} \to [0, 1]$, W is a time aggregator and I is a certainty equivalent satisfying

$$V(h) = W(h_0, I(V \circ h^1)), \text{ where } V \circ h^1 : s \in S \mapsto V(h^{s,1}). \tag{17}$$

Bommier et al. (2017) show in their Lemma 1 that a preference on a dynamic setting satisfying the axioms of weak order, continuity, recursivity, history independence and stationarity (D1–D5) can be represented by a utility function V that obeys the recursive equation in (17). The next step is to specialize this equation further. Bommier et al. (2017)'s Proposition 4 is the following:

Proposition 4.1 (Bommier–Kochov–Le Grand) *A binary relation $\succcurlyeq$ on $\mathcal{H}$ satisfies axioms D1-D7 if and only if it admits a recursive representation (V, W, I) such that either:*

1. *I is translation-invariant and $W(c, x) = u(c) + \beta x$, where $\beta \in (0, 1)$ and $u : C \to [0, 1]$ is a continuous, strictly increasing function such that $u(\underline{c}) = 0$ and $u(\bar{c}) = 1 - \beta$, or*
2. *I is translation-invariant and scale-invariant, and $W(c, x) = u(c)+b(c)x$, where $u, b : C \to [0, 1]$ are continuous functions such that $b(C) \subset (0, 1)$, the functions u and $u + b$ are strictly increasing, and $u(\underline{c}) = 0$ and $u(\bar{c}) = 1 - b(\bar{c})$.*

Proposition 4.1 shows that if the preference satisfies axioms D1-D7, then it has a recursive representation (V, W, I) where W has an additively separable structure, that is, $W(c, x) = u(c) + \beta x$ or $W(c, x) = u(c) + b(c)x$.[15] The first case implies, in particular, that the decision maker evaluates riskless consumer streams according to the standard exponential discounting utility $\sum_{t=0}^{\infty} \beta^t u(c_t)$.

4.4 Axiomatization of recursive quantile preferences

Given these results, we now adapt the static quantile axioms to a dynamic setting and combine them with all of Bommier et al. (2017)'s axioms. We wish to show that the certainty equivalent I[·] above is a quantile operator and that $W(c, x) = u(c) + \beta x$, thus obtaining the recursive equation

$$V(h) = u(h_0) + \beta \mathrm{Q}_\tau^p[V(h^1)].$$

Let $\Omega = S^\infty$, C and $\mathcal{H}$ be as in Sect. 4.1. Consider the following axioms:

Axiom A1 (*Time Separability for Deterministic Prospects*). For all $c^\infty, \hat{c}^\infty \in C^\infty$, $c, c', d, d' \in C$, we have:

$$(c, d, c^\infty) \succcurlyeq (c', d', c^\infty) \Leftrightarrow (c, d, \hat{c}^\infty) \succcurlyeq (c', d', \hat{c}^\infty).$$

Axiom A2 (*Ordinality*).
For any $c_0 \in C$, functions $f, g : S \to C$, $h = (h_0, h_1, h_2, \ldots) \in \mathcal{H}$, and increasing $\varphi : C \to C$,

$$\begin{aligned}&(c_0, f(\cdot), h_2(\cdot), \ldots) \succcurlyeq (c_0, g(\cdot), h_2(\cdot), \ldots)\\ &\Rightarrow (c_0, \varphi(f(\cdot)), h_2(\cdot), \ldots) \succcurlyeq (c_0, \varphi(g(\cdot)), h_2(\cdot), \ldots).\end{aligned}$$

Axiom A3 (*Betting Consistency*). Let $\{A_1, \ldots, A_n\} \subset 2^S$ and $\{B_1, \ldots, B_n\} \subset 2^S$ be such that $\sum_{i=1}^n 1_{A_i} \geq \sum_{i=1}^n 1_{B_i}$. Then, there exists $i \in \{1, \ldots, m\}$ such that $\mathbf{c}A_i\mathbf{c}' \succcurlyeq \mathbf{c}B_i\mathbf{c}'$, for any $\mathbf{c} = (c, c, \ldots)$ and $\mathbf{c}' = (c', c', \ldots)$ such that $c \succ c'$.

Axiom A1 is just Koopmans (1972)'s Postulate 3". Note that this is required only for deterministic prospects. This axiom allows us to show that in Proposition 4.1, $b(c) = \beta \in (0, 1)$, that is, the discounting factor is a constant and does not depend on the consumption.

Axioms A2 and A3 are adaptations of axioms Q4 and Q5, respectively, from the static to the recursive case. They allow us to show that the certainty equivalent $I[\cdot]$ in Proposition 4.1 is in fact a quantile.

We are now able to state and prove our axiomatization result:

Theorem 3 *A binary relation $\succcurlyeq$ on $\mathcal{H}$ satisfies axioms D1-D7 and A1-A3 if and only if there exist $\beta, \tau \in (0, 1)$, a continuous, strictly increasing function $u : C \to [0, 1]$*

[15] Note that D6 has a strict monotonicity requirement built into it. Indeed, a time aggregator like $W(c, x) = \min\{c, x\}$ would fail to satisfy it.

such that $u(\underline{c}) = 0$ *and* $u(\bar{c}) = 1 - \beta$, *and a probability* $p : \Sigma \to [0, 1]$ *on* $\Sigma = 2^S$ *such that* $\succcurlyeq$ *admits a recursive representation* (V, W, I), *where* $W(c, x) = u(c) + \beta x$ *and* $I : B_0(\Sigma) \to \mathbb{R}_+$ *is given by* $I(b) = \mathrm{Q}_\tau^p[b]$, *for any* $b \in B_0(\Sigma)$. *That is,* V *satisfying the recursive equation*

$$V(h) = u(h_0) + \beta \mathrm{Q}_\tau^p[V(h^1)], \tag{18}$$

represents $\succcurlyeq$, *where* $h = (h_0, h_1, \ldots) \in \mathcal{H}$ *and* $h^1 = (h_1, h_2, \ldots) \in \mathcal{H}^S$.

It is important to discuss some aspects of Theorem 3 and the representation it delivers. Ordinality and Betting Consistency, that come from A2 and A3, respectively, imply that the certainty equivalent, $I[\cdot]$ in (17) is of the quantile form. Ordinality, which is the central axiom for this result, essentially requires that only the order of the consequences (not its cardinal index) matters for the rank of uncertain prospects. This also allows to separate beliefs and tastes, as commented above in Sect. 3, in the discussion after axiom Q4. It remains only to restrict $b(c)$ in Proposition 4.1 to be constant, which is obtained with the time separability of deterministic prospects (A1). In sum, the additive separability of the recursive function W and the Ordinality of the certainty equivalent $I[\cdot]$ are characteristics that jointly lead to our model.

Using the quantile recursive equation (18) as a starting point, de Castro and Galvao (2019) develop a dynamic model of rational behavior under uncertainty, in which the agent maximizes the stream of the future τ-quantile utilities, for $\tau \in (0, 1)$. That is, the agent has a quantile preference instead of the standard expected utility. Because of recursivity, the obtained preferences are dynamically consistent. Although quantiles do not share some of the helpful properties of expectations, such as linearity and the law of iterated expectations, de Castro and Galvao (2019) establish all the standard results in dynamic models. Namely, they show that the corresponding dynamic problem yields a value function, via a fixed point argument, establish its concavity and differentiability and show that the principle of optimality holds. Additionally, they derive the corresponding Euler equation.

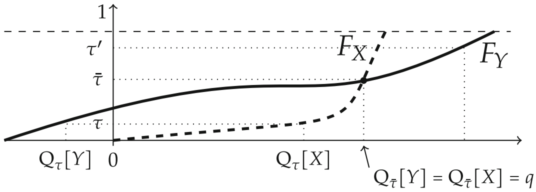
5 Risk attitude in quantile preferences

In this section, we discuss the risk attitude in quantile preferences. de Castro and Galvao (2019) also discuss the intertemporal attitudes of this preferences and show how they allow the separation of the risk attitude and the elasticity of intertemporal substitution (EIS).

5.1 Risk attitude in the static quantile model

In order to discuss the risk attitude under quantile preferences, let us introduce the concept of quantile-preserving spreads, which is related to the familiar Rothschild and Stiglitz (1970)'s mean-preserving spreads and, as this last one, captures the notion of "added noise." That is, the intuition that Y is equal to X plus noise can be formalized either as Y is a mean-preserving spread of X or that Y is a quantile-preserving spread.

Fig. 1 Y is a $\bar{\tau}$-quantile-preserving spread of X



The choice of the formalization is a subjective matter. In order to discuss this, let us follow Mendelson (1987) and define:

Definition 5.1 *(Quantile-preserving spread)* We say that Y is a τ-quantile-preserving spread of X for $q \in \mathbb{R}$, if $Q_\tau[Y] = Q_\tau[X] = q$ and the following holds:

(i) $t < q \implies F_Y(t) \geq F_X(t)$;
(ii) $t > q \implies F_Y(t) \leq F_X(t)$.

Y is a quantile-preserving spread of X if it is a $\bar{\tau}$-quantile-preserving spread of X for $F_X(q) = F_Y(q) = \bar{\tau}$, with $\bar{\tau} \in (0, 1)$.

Figure 1 illustrates the CDFs of random variables Y and X when Y is a $\bar{\tau}$-quantile-preserving spread of X. Mendelson (1987) formalizes other four conditions and show that they are all equivalent to the above definition; see this paper for further discussion and intuition. Notice that this definition captures the notion that Y is riskier than X, since it puts weight in more extreme values than X. Manski (1988) uses a different terminology for the same concept referring to the property of "single crossing from below": F_X crosses F_Y from below when Y is a quantile-preserving spread of X.

Note that if $Q_\tau[Y] = q$ and X is equal to q with probability 1, then Y is a τ-quantile-preserving spread of X. In other words, any risky asset Y with τ-quantile q is a quantile-preserving spread of any risk-free asset X with value q.

Figure 1 suggests that the choice of a τ-quantile maximizer or τ-decision maker (τ-DM) depends on whether τ is below or above the quantile $\bar{\tau}$ where the two CDFs cross. That is, when $\tau < \bar{\tau}$ as in Fig. 1, a τ-DM prefers the safer asset X: $Q_\tau[X] \geq Q_\tau[Y]$. On the other hand, if $\tau' > \bar{\tau}$, a τ-DM prefers the riskier asset Y: $Q_\tau[X] \leq Q_\tau[Y]$. The following result formalizes this intuition.

Proposition 5.2 (Manski) *Let Y be a $\bar{\tau}$-quantile-preserving spread of X for $\bar{\tau} \in (0, 1)$. Then:*

(i) $\tau \leq \bar{\tau} \implies Q_\tau[X] \geq Q_\tau[Y]$, *that is, a τ-DM prefers the less risky asset X if τ is low;*
(ii) $\tau \geq \bar{\tau} \implies Q_\tau[X] \leq Q_\tau[Y]$, *that is, a τ-DM prefers the riskier asset Y if τ is high.*

We turn now to the problem of comparing the risk attitude of τ-DM with different τ. For this, consider the following definition, due to Epstein (1999, equation (2.1), p. 583):

Definition 5.3 (Epstein 1999) Preference $\succcurlyeq^1$ is more risk averse than preference $\succcurlyeq^2$ if for all $x \in \Upsilon$, and $f \in \mathcal{F}$, $x \succcurlyeq^2 f \Rightarrow x \succcurlyeq^1 f$ and $x \succ^2 f \Rightarrow x \succ^1 f$.

The intuition for this definition is clear given that constant acts do not present risk or are perfectly certain: if $\succcurlyeq^2$ prefers safety, the more risk averse $\succcurlyeq^1$ would prefer too. We can construct a similar characterization of risk in the quantile setting that allows one to rank preferences. Let $\succcurlyeq^\tau$ represent a τ-quantile preference. The following formalizes the result.

Proposition 5.4 *The following statements are equivalent:*

1. $\tau_1 < \tau_2$;
2. $\succcurlyeq^{\tau_1}$ *is more risk averse than* $\succcurlyeq^{\tau_2}$;
3. *If Y is a quantile-preserving spread of X and a τ_2-DM prefers X to Y, then so does a τ_1-DM.*[16]
4. *If Y is a quantile-preserving spread of X and a τ_1-DM prefers Y to X, then so does a τ_2-DM.*

This result shows that $\succcurlyeq^{\tau_1}$ is more risk averse than $\succcurlyeq^{\tau_2}$ if and only if $\tau_1 < \tau_2$. This property implies that an agent with quantile given by τ_1 is more risk preferring than another agent with quantile given by τ_2 if $\tau_1 > \tau_2$, independently of the functional form of the utility function. Thus, a decision maker that maximizes a lower quantile is more risk averse than one who maximizes a higher quantile. In other words, the risk attitude can be related to the quantile rather than to the concavity of the utility function. Moreover, Proposition 5.4 shows that in the QU framework individuals' risk attitude is related to the quantile rather than to the concavity of the utility function.

5.2 Risk attitude in the dynamic quantile model

Now we discuss the notion of risk attitude in the dynamic quantile model. For this discussion, consider preferences $\succcurlyeq^i$, $i \in \{1, 2\}$, satisfying axioms D1–D7 and A1, so that they are represented by V^i satisfying the following recursive equation:

$$V^i(h) = u(h_0) + \beta I^i[V^i(h^1)].$$

As Epstein and Zin (1989), we adapt Definition 5.3 for the dynamic case as follows: we say that $\succcurlyeq^1$ is more uncertainty averse than $\succcurlyeq^2$ if, for all $c^\infty \in C^\infty$ and $h \in \mathcal{H}$,

$$c^\infty \succcurlyeq^2 h \implies c^\infty \succcurlyeq^1 h. \tag{19}$$

We have the following:

Proposition 5.5 $\succcurlyeq^1$ *is more risk averse than* $\succcurlyeq^2$ *if and only if* $I^1[\cdot] \leq I^2[\cdot]$.

Specializing the above result to the quantile case, we conclude that $\succcurlyeq^1$ is more risk averse than $\succcurlyeq^2$ if and only if $I^1(\cdot) = \mathrm{Q}_{\tau_1}[\cdot] \leq \mathrm{Q}_{\tau_2}[\cdot] = I^2(\cdot)$ which is equivalent to $\tau_1 \leq \tau_2$. This is exactly the notion obtained in Proposition 5.4 above. Hence, as in Manski (1988) and Rostek (2010), the dynamic quantile model admits a notion of

[16] Notice that we are not specifying what is the quantile $\bar{\tau}$ for which Y is a $\bar{\tau}$-quantile-preserving spread of X. The same observation is valid for item 4.

comparative risk attitude, where τ captures a measure of risk attitude, but agents are not characterized as risk-averse, risk-neutral, or risk-seeking. Moreover, this definition of risk allows for the risk attitudes to be disentangled from the degree of intertemporal substitutability, as we discuss next.

In summary, the single dimensional parameter $\tau \in (0, 1)$ captures the risk attitude in quantile models. This is in contrast to the risk attitude in expected utility model, which is defined by the entire utility function, which is an object that usually belongs to infinite dimensional spaces. This remains true if one tries to characterize the risk attitude through the coefficient of relative or absolute risk aversion, which are also functions. Only when we further particularize the expected utility models to constant coefficients (CRRA and CARA), we are able to obtain a single dimensional parameter pinning down the risk attitude. Interestingly, this is what is used in practice. In quantile models this particularization is not needed, since the model is already unidimensional and, thus, simpler.

6 Conclusion and open questions

This paper axiomatizes static and dynamic quantile preferences. A central objective is to establish foundations for the dynamic preference. There are previous axiomatizations for the static case, such as Chambers (2009) and Rostek (2010), but the methods used to derive the dynamic representation preclude us from using those previous results. We cannot use Chambers (2009) because he works in a setting of risk, instead of uncertainty, while Rostek (2010) requires an infinite state space, which is not convenient for the axiomatization of the dynamic preferences.

We first axiomatize quantile preferences for the static case with finite state space. The main axioms that provide the quantile preferences representation in the static case are Monotonicity, Ordinality, and Betting Consistency. Second, we develop an axiomatization of the recursive quantile preferences. For this, we apply the results in Bommier et al. (2017) and extend the Monotonicity, Ordinality, and Betting Consistency axioms from the static to the dynamic context. The dynamic preferences induce an additively separable quantile model with standard discounting, that is, the recursive equation is characterized by the sum of the current period utility function and the discounted value of the certainty equivalent, which is a quantile function. These preferences are time consistent and have a simple quantile recursive representation, which gives the model the analytical tractability needed in several fields in financial and economic applications. Finally, we study the notion of risk attitude in both the static and recursive quantile models. In quantile models, the risk attitude is completely captured by the quantile τ, a single-dimensional parameter. This is simpler than in expected utility models, where in general the risk attitude is determined by a function.

It is left open the extension of the dynamic preference axiomatization to infinite state spaces, which would probably also require to extend Bommier et al. (2017)'s results. Another direction to explore is whether there are preferences distinct from quantile preferences that obey Ordinality (Q4) and perhaps some variations of some of the other axioms (Q1, Q2, Q3 and Q5).

A Appendix

We need to introduce some notation. If $x \sim y$ for all $x, y \in \Upsilon$, this would contradict Q1. Therefore, from now on, we assume that there exists $\overline{x}$ and $\underline{x} \in \Upsilon$, such that $\overline{x} \succ \underline{x}$. For the rest of the proof, let this $\overline{x}$ and $\underline{x}$ be fixed. We begin with the following auxiliary result:

Lemma A.1 *Assume Q1, Q4 and Q2 in the Anscombe–Aumann setting or Q2'in the topological setting. For any $x, y \in \Upsilon$ and any $E \subset S$, we have $xEy \sim x$ or $xEy \sim y$.*

Proof If $x \sim y$, there is nothing to prove, since this and monotonicity would imply $xEy \sim x \sim y$. Thus, let us assume, without loss of generality, that $x \succ y$. By monotonicity, $x \succcurlyeq xEy \succcurlyeq y$. Assume that $xEy \succ y$. We claim that there exists $z \in \Upsilon$ such that $z \sim xEy$.

To see the validity of the claim, consider first the Anscombe–Aummann setting. Consider the sets $A \equiv \{\alpha \in [0, 1] : \alpha x + (1 - \alpha)y \succcurlyeq xEy\}$ and $B \equiv \{\alpha \in [0, 1] : xEy \succcurlyeq \alpha x + (1 - \alpha)y\}$. Then, $1 \in A$, $0 \in B$, A and B are closed by continuity (Q2) and $A \cup B = [0, 1]$ by completeness. By the connectedness of $[0, 1]$, there exists $\alpha \in A \cap B$. Since Υ is convex, $z = \alpha x + (1 - \alpha)y \in \Upsilon$ and $z \sim xEy$. In the topological setting, consider the sets $A \equiv \{f \in \mathcal{F} : f \succcurlyeq xEy\} \cap \Upsilon$ and $B \equiv \{f \in \mathcal{F} : xEy \succcurlyeq f\} \cap \Upsilon$, where Υ is seen as a subset of $\mathcal{F}$, as usual. Since $x \in A$ and $y \in B$, $A \cup B = \Upsilon$, which is also connected, the claim follows.

Let $z \in \Upsilon$ be such that $z \sim xEy$ as above. Define $\varphi : \Upsilon \to \Upsilon$ by:

$$\varphi(w) = \begin{cases} x, & \text{if } w \succ y \\ y, & \text{if } w \preccurlyeq y \end{cases}$$

Clearly, φ is increasing. Since $z \sim xEy \succ y$, $\varphi(z) = x$. Since $xEy \sim z$, by Ordinality we have $xEy = \varphi(xEy) \sim \varphi(z) = x$. This concludes the proof. □

We will prove first Theorem 2, which is inspired in Chambers (2007). The proof of Theorem 1 will be discussed in the sequel. A collection of sets $\mathcal{E} \subset \Sigma$ is a downset if $A \in \mathcal{E}$ and $B \subset A$ and $B \in \Sigma$ implies that $B \in \mathcal{E}$.

Proposition A.2 *If $\succcurlyeq$ satisfies axioms Q1, Q2' and Q4, there exists $u : \Upsilon \to \mathbb{R}$ and a unique downset $\mathcal{E} \subset \Sigma$ such that $\emptyset \in \mathcal{E}$ and $S \notin \mathcal{E}$ for which*

$$U(f) = \inf\{\alpha : \{s \in S : u(f(s)) \geq \alpha\} \in \mathcal{E}\}.$$

Proof From Q1 and Q2', there exists a continuous function $U : \mathcal{F} \to \mathbb{R}$ that represents $\succcurlyeq$. Since we see Υ as a subset of $\mathcal{F}$, we can define $u : \Upsilon \to \mathbb{R}$ by $u(x) = U(x)$. Without loss of generality, we may assume that $u(\overline{x}) = 1$ and $u(\underline{x}) = 0$. For notation simplicity, in this proof only we denote by 1_E the act that is equal to $\overline{x}$ if $s \in E$ and is equal to $\underline{x}$ if $s \notin E$. By Lemma A.1, we have $U(1_E) \in \{0, 1\}$ for all $E \subset S$.

Let us define $\mathcal{E}$ as the set of those $E \in \Sigma$ such that $U(1_E) = 0$. It is easy to see that this defines a downset: if $B \subset E$, $B \in \Sigma$, $E \in \mathcal{E}$ then $1_E \succcurlyeq 1_B$ by monotonicity. Therefore, by Lemma A.1, $0 = U(1_E) \geq U(1_B) \in \{0, 1\}$, which implies $U(1_B) =$

$0 \Rightarrow B \in \mathcal{E}$. Moreover, $S \notin \mathcal{E}$ since $\overline{x} = \overline{x}S\underline{x}$ and, therefore, $U(1_S) = u(\overline{x}) = 1$. Analogously, $\varnothing \in \mathcal{E}$.

We will show that for all $E \in \Sigma$, $U(1_E) = \inf\{\alpha \in [0, 1] : \{s \in S : u(1_E(s)) \geq \alpha\} \in \mathcal{E}\}$. Consider first the case $E \in \mathcal{E}$. For all $\alpha \in (0, 1)$, $\{s \in S : u(1_E(s)) \geq \alpha\} = E$, so that $\{s \in S : u(1_E(s)) \geq \alpha\} \in \mathcal{E}$. Therefore, $\inf\{\alpha : \{s \in S : u(1_E(s)) \geq \alpha\} \in \mathcal{E}\} \leq 0$. However, for all $\alpha < 0$, $\{s \in S : 1_E(s) \geq \alpha\} = S \notin \mathcal{E}$. Hence, we may conclude that $\inf\{\alpha : \{s \in S : u(1_E(s)) \geq \alpha\} \in \mathcal{E}\} = 0$, so that $U(1_E) = \inf\{\alpha : \{s \in S : u(1_E(s)) \geq \alpha\} \in \mathcal{E}\}$.

Suppose now that $E \notin \mathcal{E}$. Then for all $\alpha > 1$, $\{s \in S : u(1_E(s)) \geq \alpha\} = \emptyset$, so that $\inf\{\alpha : \{s \in S : u(1_E(s)) \geq \alpha\} \in \mathcal{E}\} \leq 1$. But for $\alpha \in (0, 1)$, $\{s \in S : u(1_E(s)) \geq \alpha\} = E$, so that $\{s \in S : u(1_E(s)) \geq \alpha\} \notin \mathcal{E}$. Hence, $\inf\{\alpha : \{s \in S : u(1_E(s)) \geq \alpha\} \in \mathcal{E}\} = 1$. Therefore, $U(1_E) = \inf\{\alpha : \{s \in S : u(1_E(s)) \geq \alpha\} \in \mathcal{E}\}$. This also shows that $\mathcal{E}$ is the unique downset that can satisfy (10) for acts 1_E. Moreover, for any act xEy, with $x \succ y$, we have $xEy = \varphi(1_E)$ for some increasing φ. Thus, Lemma A.1 and Ordinality imply that

$$U(xEy) = u(x),\ if\ E \notin \mathcal{E}\ and\ U(xEy) = u(y)\ if\ E \in \mathcal{E}. \tag{20}$$

Next, we extend the result to all functions in $\mathcal{F}$. Let $f \in \mathcal{F}$ be arbitrary, and set $\alpha^*(f) = \inf\{\alpha : \{s \in S : u(f(s)) \geq \alpha\} \in \mathcal{E}\}$. We want to conclude that $U(f) \leq \alpha^*(f)$. Let $\epsilon > 0$. Then $\{s \in S : u(f(s)) \geq \alpha^*(f) + \epsilon\} \in \mathcal{E}$ by definition of $\alpha^*(f)$. Let $g^\epsilon \in \mathcal{F}$ be defined by xEy, where $E = \{s \in S : u(f(s)) \geq \alpha^*(f) + \epsilon\}$, and $x, y \in \Upsilon$ are any consequences such that $x \succcurlyeq f(s), \forall s \in E$ and $y \succcurlyeq f(s), \forall s \notin E$, so that $u(y) \leq \alpha^*(f) + \epsilon$. These x and y exist since S is finite. Thus, $g^\epsilon(s) \succcurlyeq f(s), \forall s \in S$, which implies, by monotonicity, $U(f) \leq U(g^\epsilon)$. By Lemma A.1 and (20), $U(g^\epsilon) = u(y) \leq \alpha^*(f) + \epsilon$. Since ϵ is arbitrary, $U(f) \leq \alpha^*(f)$.

Now we wish to conclude that $U(f) \geq \alpha^*(f)$.

Let $\epsilon > 0$. Then $\{s \in S : u(f(s)) \geq \alpha^*(f) - \epsilon\} \notin \mathcal{E}$ by definition of $\alpha^*(f)$. Let $h^\epsilon \in \mathcal{F}$ be the act xEy where $E = \{s \in S : u(f(s)) \geq \alpha^*(f) - \epsilon\}$, and $x, y \in \Upsilon$ are any consequences such that $f(s) \succcurlyeq x, \forall s \in E$, $f(s) \succcurlyeq y, \forall s \notin E$ and $u(x) \geq \alpha^*(f) - \epsilon$. This implies that $f(s) \succcurlyeq h^\epsilon(s), \forall s \in S$ and $U(f) \geq U(h^\epsilon)$. Moreover, $\{s \in S : u(f(s)) \geq \alpha^*(f) - \epsilon\} \notin \mathcal{E}$. By (20), $U(h^\epsilon) = u(x) \geq \alpha^*(f) - \epsilon$. Thus, $U(f) \geq \alpha^*(f) - \epsilon$. As ϵ is arbitrary, $U(f) \geq \alpha^*(f)$. Therefore $U(f) = \inf\{\alpha : \{s \in S : u(f(s)) \geq \alpha\} \in \mathcal{E}\}$. □

Proof of Sufficiency in Theorem 2 This proof adapts the argument of Chambers (2007, Theorem 2) to exclude the case $\tau = 0$.[17] We reproduce the whole argument here for completeness and readers' convenience. Let $U : \mathcal{F} \to \mathbb{R}$ and $\mathcal{E}$ be respectively the utility function and the downset shown to exist by Proposition A.2. Let $S = \{s_1, \ldots, s_n\}$ and for any $E \subset S$, let $\bar{u}^E$ denote the vector in $\{0, 1\}^n$ such that $\bar{u}^E_i = 1$ if and only if $s_i \in E$.

We want to show that there exists a probability measure represented by a vector $p \in [0, 1]^n$, satisfying $\sum_{i=1}^n p_i = 1$, and a number $\tau \in (0, 1)$ such that $\mathcal{E} = \{E \in \Sigma : p(E) \leq \tau\} = \{E \in \Sigma : p \cdot \bar{u}^E \leq \tau\}$. For this, we claim that it is sufficient to show

[17] His proof quotes Rockafellar (1970, Theorem 22.2) that does not deliver the conclusion needed. We provide another reference in the proof below and spell out all details of the argument.

that there exists $(p, \tau) \in \mathbb{R}^{n+1}$ solution of the following system of linear inequalities:

$$(\bar{u}^E, -1) \cdot (p, \tau) > 0, \forall E \notin \mathcal{E}; \tag{21}$$

$$(-\bar{u}^E, 1) \cdot (p, \tau) \geq 0, \forall E \in \mathcal{E}; \tag{22}$$

$$(\bar{u}^{\{s\}}, 0) \cdot (p, \tau) \geq 0, \forall s \in S. \tag{23}$$

To see the claim, assume that there exists $(\tilde{p}, \tilde{\tau}) \in \mathbb{R}_+^{n+1}$ satisfying (21)–(23). By Proposition A.2, we know that $\varnothing \in \mathcal{E}$ and $S \notin \mathcal{E}$. Therefore, (21) and (22) imply that:

$$0 = \tilde{p} \cdot \bar{u}^\varnothing \leq \tilde{\tau} < \tilde{p} \cdot \bar{u}^S.$$

By (23), we know that $\tilde{p}_i \geq 0$ for all $i = 1, \ldots, n$. Therefore, we can define $(p, \tau) \equiv \frac{1}{\tilde{p} \cdot \bar{u}^S}(\tilde{p}, \tilde{\tau})$ and observe that $(p, \tau) \in [0, 1]^{n+1}$ is such that $\tau < p \cdot \bar{u}^S = 1$, that is, p is a probability measure, $\tau \in [0, 1)$ and $\mathcal{E} = \{E \in \Sigma : p(E) \leq \tau\}$, because of (21) and (22). It remains to show that we can choose $\tau > 0$. Indeed, if $\tau = 0$, let $S^0 \equiv \{s \in S : p(\{s\}) = 0\}$ and fix $\hat{s} \in S \backslash S^0$ such that $p(\{\hat{s}\}) = \min\{p(\{s\}) : s \in S \backslash S^0\} > 0$. Let $n^0 \equiv |S^0|$ and $\epsilon \equiv p(\{\hat{s}\})$. Define $\hat{\tau} = \frac{\epsilon}{2}$ and for each $s \in S$,

$$\hat{p}(\{s\}) = \begin{cases} p(\{s\}), & if\ s \notin S^0 \cup \{\hat{s}\} \\ \frac{2\epsilon}{3}, & if\ s = \hat{s} \\ \frac{\epsilon}{3n^0}, & if\ s \in S^0 \end{cases}$$

Then, $\hat{p}$ is a probability, $\hat{\tau} \in (0, 1)$ and $\mathcal{E} = \{E \in \Sigma : p(E) \leq \tau\} = \{E \in \Sigma : \hat{p}(E) \leq \hat{\tau}\}$.

Therefore, the sufficiency of Theorem 2 is established if we show that the system of inequalities (21)–(23) has a solution (p, τ). For this, we use a theorem of the alternative for matrices with rational entries; see for instance Fishburn (1986, Lemma 1, p. 234). We state the theorem here for readers' convenience. □

Theorem 4 *Let $a^i \in \mathbb{R}^N$ have rational components for $i = 1, \ldots, m$, and let $k \in \{1, \ldots, m\}$. Then one and only one of the following alternatives holds:*

(a) *There exists a vector $x \in \mathbb{R}^N$ such that*

$$\begin{aligned} a^i \cdot x > 0, &\qquad for\ i = 1, \ldots, k \\ a^i \cdot x \geq 0, &\qquad for\ i = k+1, \ldots, m \end{aligned}$$

(b) *There exist nonnegative integers $r_1, \ldots, r_m$ such that $r_i > 0$ for at least one $i \in \{1, \ldots, k\}$ and*

$$\sum_{i=1}^{m} r_i a^i_j = 0\ for\ j = 1, \ldots, N.$$

Suppose, for a contradiction, that the system of inequalities (21)–(23) does not have a solution $x = (p, \tau) \in \mathbb{R}^{n+1}$. Let $N \equiv n+1$, $m \equiv n + 2^n$ and $k \equiv |\mathcal{E}^c| = 2^n - |\mathcal{E}|$. Enumerate the subsets of S as $E_1, \ldots, E_{2^n}$ in such a way that the first k sets are not members of $\mathcal{E}$, that is, $E_i \in \mathcal{E} \Leftrightarrow i > k$. Define a^i as follows:

$$a^i \equiv \begin{cases} (\bar{u}^{E_i}, -1), & \text{if } i = 1, \ldots, k; \\ (-\bar{u}^{E_i}, 1), & \text{if } i = k+1, \ldots, m-n; \\ (\bar{u}^{\{s_{i-m+n}\}}, 0), & \text{if } i = m-n+1, \ldots, m. \end{cases}$$

Note that all a^i's have integer components. By Theorem 4, we conclude that there exist nonnegative integers $r_1, \ldots, r_m$ such that $r_i > 0$ for at least one $i \in \{1, \ldots, k\}$ and

$$\sum_{i=1}^{m} r_i a^i_j = 0 \text{ for } j = 1, \ldots, N. \tag{24}$$

Taking $j = N = n+1$ above, we conclude that

$$-\sum_{i=1}^{k} r_i + \sum_{i=k+1}^{m-n} r_i = 0 \Leftrightarrow r \equiv \sum_{i=1}^{k} r_i = \sum_{i=k+1}^{m-n} r_i. \tag{25}$$

Since $r_i > 0$ for at least one $i \in \{1, \ldots, k\}$, we have $r > 0$. For each $j \in \{1, \ldots, n\}$, define the sets $N_j \equiv \{\ell \in \{1, \ldots, k\} : s_j \in E_\ell\}$ and $M_j \equiv \{\ell \in \{k+1, \ldots, m-n\} : s_j \in E_\ell\}$. Thus, for $j = 1, \ldots, n$ in (24), we obtain:

$$r_{j+m-n} + \sum_{\ell \in N_j} r_\ell = \sum_{\ell \in M_j} r_\ell \Rightarrow \sum_{\ell \in N_j} r_\ell \leq \sum_{\ell \in M_j} r_\ell. \tag{26}$$

Now, we will define two lists of sets $A_t, B_t \subset S$, both of the same length $r \geq 1$. We define the sets $A_1, \ldots, A_{r_{k+1}}$ to be all equal to E_{k+1}; the sets $A_{r_{k+1}+1}, \ldots, A_{r_{k+1}+r_{k+2}}$ to be all equal to E_{k+2}, and we continue in this way until we have defined all $r = \sum_{i=k+1}^{m-n} r_i$ of the A_t sets. Analogously, the sets $B_1, \ldots, B_{r_1}$ are all equal to E_1; the sets $B_{r_1+1}, \ldots, B_{r_1+r_2}$ are equal to E_2, and we continue in this fashion, until we have defined all $r = \sum_{i=1}^{k} r_i$ of the B_t sets. Observe that $A_t \in \mathcal{E}$ and $B_t \notin \mathcal{E}$ for all $t \in \{1, \ldots, r\}$. By the Proposition A.2, $U(1_{A_t}) = 0$ and $U(1_{B_t}) = 1$, that is, $\overline{x}\, B_t\, \underline{x} \succ \overline{x}\, A_t\, \underline{x}$, $\forall t \in \{1, \ldots, r\}$. Now, observe that for each $s_j \in S$, by (26),

$$\sum_{t=1}^{r} 1_{A_t}(s_j) = \sum_{\ell \in M_j} r_\ell \geq \sum_{\ell \in N_j} r_\ell = \sum_{t=1}^{r} 1_{B_t}(s_j).$$

Therefore, we obtain a contradiction of Betting Consistency (Q5). $\square$

For the proof necessity in Theorem 2, it is convenient to introduce some notation. For any $f \in \mathcal{F}$, we denote the image of f by Υ^f, that is, $\Upsilon^f \equiv f(S) = \{x^f_1, \ldots, x^f_n\}$.

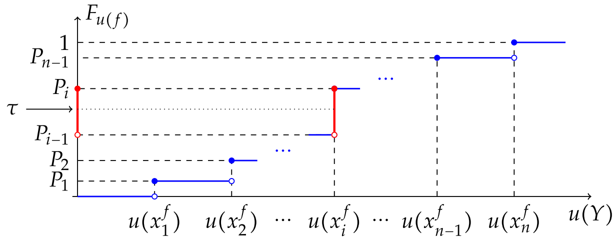


Fig. 2 c.d.f. of $u(f)$ and quantile $Q_\tau^p[u(f)] = u(x_i^f)$

Without loss of generality, we may assume $u(x_1^f) < \cdots < u(x_n^f)$. Let $E_i^f \equiv \{s \in S : f(s) = x_i^f\}$ and $P_i^f \equiv \sum_{j=1}^i p(E_j^f)$. This is illustrated in Fig. 2 below, omitting the superscript in P_i^f. With this convention, we have:

$$U(f) = Q_\tau^p[u(f)] = u(x_i^f), \text{ if } \tau \in (P_{i-1}^f, P_i^f]. \tag{27}$$

As part of the proof the necessity of Theorem 2, we need to establish that Q4 holds for the quantile representation. This was separately stated as Proposition 3.8. Thus, we first prove it.

Proof of Proposition 3.8 Consider $f, g \in \mathcal{F}$ and an increasing $\varphi : \Upsilon \to \Upsilon$. We want to show that

$$Q_\tau^p[u(f)] \geq Q_\tau^p[u(g)] \implies Q_\tau^p[u(\varphi(f))] \geq Q_\tau^p[u(\varphi(g))].$$

As above, let $\Upsilon^f \equiv f(S) = \{x_1^f, \ldots, x_n^f\}$ and $\Upsilon^g \equiv g(S) = \{x_1^g, \ldots, x_m^g\}$, with $u(x_1^f) < \cdots < u(x_n^f)$ and $u(x_1^g) < \cdots < u(x_n^g)$. Let $U(f) = u(x_i^f) \Leftrightarrow \tau \in (P_{i-1}^f, P_i^f]$ and $U(g) = u(x_j^g) \Leftrightarrow \tau \in (P_{j-1}^g, P_j^g]$. We assume that $Q_\tau^p[u(f)] \geq Q_\tau^p[u(g)]$, that is, $u(x_i^f) \geq u(x_j^g)$. Of course $Q_\tau^p[u(\varphi(f))] \in \{u(\varphi(x_1^f)), \ldots, u(\varphi(x_i^f)), \ldots, u(\varphi(x_n^f))\}$ and $u(\varphi(x_1^f)) \leq \cdots \leq u(\varphi(x_n^f))$. Equalities may occur because φ is only increasing. Such equalities may change the index of x. that defines the quantiles. However, we can see that if $Q_\tau^p[u(\varphi(f))] = u(\varphi(x_k^f))$ then $u(\varphi(x_k^f)) = u(\varphi(x_{k+1}^f)) = \cdots = u(\varphi(x_i^f))$. Similarly, $Q_\tau^p[u(\varphi(g))] = u(\varphi(x_\ell^g))$ implies $u(\varphi(x_\ell^g)) = u(\varphi(x_{\ell+1}^g)) = \cdots = u(\varphi(x_j^g))$. Since $u(x_j^g) \leq u(x_i^f)$, $u(\varphi(x_j^g)) \leq u(\varphi(x_i^f))$. Therefore, $u(\varphi(x_\ell^g)) \leq u(\varphi(x_k^f))$. Thus, $Q_\tau^p[u(\varphi(f))] \geq Q_\tau^p[u(\varphi(g))]$, as we wanted to show. □

Proof of Necessity in Theorem 2 Assume that $U(f) = Q_\tau^p[u(f)]$ represents $\succcurlyeq$, for some nonconstant continuous utility $u : \Upsilon \to \mathbb{R}$, probability $p : \Sigma \to [0, 1]$ and $\tau \in (0, 1)$, where

$$Q_\tau^p[u(f)] \equiv \inf\{\alpha : p(\{s \in S : u(f(s)) \geq \alpha\}) \leq \tau\}.$$

Axiom Q1 is easily seen to be satisfied: monotonicity follows the well-known property of monotonicity of quantiles (see, for instance, de Castro and Galvao (2019, Lemma A.1(vi), p. 1926)); nontriviality follows from the fact that u is nonconstant.

For Q2'(Continuity), use the terminology introduced above to observe that $\{g \in \mathcal{F} : f \succcurlyeq g\} = \{g \in \mathcal{F} : u(x_j^g) \leq u(x_i^f)\}$, which is easily seen to be closed in $\mathcal{F}$. Analogous reasoning works for $\{g \in \mathcal{F} : g \succcurlyeq f\}$, which establishes Q2'.

To see that Q4 (Ordinality) is satisfied, it is sufficient to observe that $f \succcurlyeq g \Leftrightarrow \mathrm{Q}_\tau^p[u(f)] \geq \mathrm{Q}_\tau^p[u(g)]$ and $\varphi(f) \succcurlyeq \varphi(g) \Leftrightarrow \mathrm{Q}_\tau^p[u(\varphi(f))] \geq \mathrm{Q}_\tau^p[u(\varphi(g))]$, and apply Proposition 3.8.

To verify Betting Consistency (Q5), first observe that for any $x, y \in \Upsilon$ and $E \in \Sigma$, if $x \succ y$, then

$$U(xEy) = \begin{cases} u(x), & \text{if } \tau > 1 - p(E) \\ u(y), & \text{if } \tau \leq 1 - p(E) \end{cases} \tag{28}$$

Let $\{A_1, \ldots, A_n\} \subset 2^S$ and $\{B_1, \ldots, B_n\} \subset 2^S$ for which $\sum_{i=1}^n 1_{A_i} \geq \sum_{i=1}^n 1_{B_i}$. Suppose, for a contradiction, that for all $i \in \{1, \ldots, n\}$, $U(\overline{x} B_i \underline{x}) > U(\overline{x} A_i \underline{x})$ for given $\underline{x}, \overline{x} \in \Upsilon$, with $\overline{x} \succ \underline{x}$. This is only possible if for all $i \in \{1, \ldots, n\}$, $U(\overline{x} B_i \underline{x}) = u(\overline{x})$ and $U(\overline{x} A_i \underline{x}) = u(\underline{x})$. Hence, from (28), for all $i \in \{1, \ldots, n\}$, $p(B_i) > 1 - \tau$ and $p(A_i) \leq 1 - \tau$. Let $E_p[\cdot]$ denote the expectation with respect to p. As $\sum_{i=1}^n 1_{A_i}(s) \geq \sum_{i=1}^n 1_{B_i}(s)$, for all $s \in S$, $E_p[\sum_{i=1}^n 1_{A_i}] = \sum_{i=1}^n p(A_i) \geq E_p[\sum_{i=1}^n 1_{B_i}] = \sum_{i=1}^n p(B_i)$. However, $\sum_{i=1}^n p(B_i) > n(1-\tau) \geq \sum_{i=1}^n p(A_i)$, a contradiction. □

Now, we can adapt Proposition A.2 to the setting of Sect. 3.1.

Proposition A.3 *If $\succcurlyeq$ satisfies axioms Q1-Q4, there exists $u : \Upsilon \to \mathbb{R}$ and a unique downset $\mathcal{E} \subset \Sigma$ such that $\emptyset \in \mathcal{E}$ and $S \notin \mathcal{E}$ for which*

$$f \succcurlyeq g \iff I[u \circ f] \geq I[u \circ g],$$

where $I : B_0(\Sigma, u(\Upsilon)) \to \mathbb{R}$ is given by

$$I[\xi] = \inf\{\alpha : \{s \in S : \xi(s) \geq \alpha\} \in \mathcal{E}\}$$

for any $\xi \in B_0(\Sigma, u(\Upsilon))$.

Proof Let I and u be those given by Lemma 3.1. Without loss of generality, we may assume that $u(\overline{x}) = 1$ and $u(\underline{x}) = 0$. By Lemma A.1, we have $I[1_E] \in \{0, 1\}$. The reminder of the proof of this proposition is very similar to the proof of Proposition A.2 and we thus omit it. □

Proof of Theorem 1 With Proposition A.3, the proof of sufficiency of Theorem 1 follows closely that of Theorem 2 given above. In order to avoid repetitions, we will omit the rest the proof.

For necessity, the proofs for Q4 and Q5 are identical to those of Theorem 2. Q1 and Q2 are also easily seen to hold. For Q3, it is sufficient to recall that u is affine and that

$$\mathrm{Q}_\tau^p[u(\alpha f + (1-\alpha)x)] = \mathrm{Q}_\tau^p[\alpha u(f) + (1-\alpha)u(x)] = \alpha \mathrm{Q}_\tau^p[u(f)] + (1-\alpha)\mathrm{Q}_\tau^p[u(x)],$$

for $\alpha \in [0,1]$, $f \in \mathcal{F}$ and $x \in \Upsilon$. The property follows immediately. □

We prove Proposition 3.10 through a series of lemmas. For this, it will be useful to introduce some additional notation. In what follows, let $\varphi : \Upsilon \to \Upsilon$, an increasing function, be fixed. For any function $f : S \to \Upsilon$, let

$$\alpha_f \equiv \mathrm{Q}_\tau^p[u(f)] = \inf\{\alpha \in \mathbb{R} : p\left(\{s \in S : u(f(s)) \leq \alpha\}\right) \geq \tau\}, \tag{29}$$

for a given probability $p : \Sigma \to [0,1]$. We will also use α_g, $\alpha_{\varphi(f)}$ and $\alpha_{\varphi(g)}$ with the corresponding functions in the place of f above. Define

$$\psi(\alpha) \equiv \inf\{u(\varphi(x)) : u(x) \geq \alpha\} \tag{30}$$

and

$$\phi(\alpha) \equiv \sup\{u(\varphi(x)) : u(x) < \alpha\}. \tag{31}$$

It is clear that $\phi(\alpha) \leq \psi(\alpha)$: if $u(x) < \alpha \leq u(x')$ then $u(\varphi(x)) \leq u(\varphi(x'))$. Moreover, a known property of quantiles,[18] we have:

$$\begin{aligned} &p(\{s \in S : u(f(s)) \leq \alpha_f\}) \geq \tau; \text{ and} \\ &p(\{s \in S : u(\varphi(f(s))) \leq \alpha_{\varphi(f)}\}) \geq \tau, \end{aligned} \tag{32}$$

and similarly for g and $\varphi(g)$.

The following result will be useful below:

Lemma A.4 *The functions ψ and ϕ defined in (30) and (31), respectively, are increasing.*

Proof Let $\alpha \leq \alpha'$. Then, $\{x \in \Upsilon : u(x) \geq \alpha\} \supset \{x \in \Upsilon : u(x) \geq \alpha'\}$, which implies

$$\psi(\alpha) = \inf\{u(\varphi(x)) : u(x) \geq \alpha\} \leq \inf\{u(\varphi(x)) : u(x) \geq \alpha'\} = \psi(\alpha').$$

Similarly, $\{x \in \Upsilon : u(x) < \alpha\} \subset \{x \in \Upsilon : u(x) < \alpha'\}$, which implies

$$\phi(\alpha) = \sup\{u(\varphi(x)) : u(x) < \alpha\} \leq \sup\{u(\varphi(x)) : u(x) < \alpha'\} = \phi(\alpha').$$

□

[18] See, for instance, de Castro and Galvao (2019, Lemma A.1(iv), p. 1926).

Lemma A.5 *If* $p\left(\{s \in S : u(f(s)) < \alpha_f\}\right) \geq \tau$*, then for any* $\alpha < \alpha_f$ *there exists* $x \in \Upsilon$ *such that* $\alpha < u(x) < \alpha_f$*.*

Proof From the assumption, the definition of α_f and $\alpha < \alpha_f$, we obtain

$$\begin{aligned} & p\left(\{s \in S : u(f(s)) < \alpha_f\}\right) \geq \tau > p\left(\{s \in S : u(f(s)) \leq \alpha\}\right) \\ & \quad \Rightarrow p\left(\{s \in S : \alpha < u(f(s)) < \alpha_f\}\right) > 0. \end{aligned}$$

Thus, there exists $x \in \Upsilon$ such that $\alpha < u(x) < \alpha_f$. □

Proposition A.6 *The following inequalities hold:*

$$\phi(\alpha_f) \leq \mathrm{Q}_\tau^p[u(\varphi(f))] \leq \psi(\alpha_f). \tag{33}$$

Moreover, one of the inequalities holds as an equality. More precisely,

$$p\left(\{s \in S : u(f(s)) < \alpha_f\}\right) \geq \tau \implies \mathrm{Q}_\tau^p[u(\varphi(f))] = \phi(\alpha_f); \text{ and} \tag{34}$$

$$p\left(\{s \in S : u(f(s)) < \alpha_f\}\right) < \tau \implies \mathrm{Q}_\tau^p[u(\varphi(f))] = \psi(\alpha_f). \tag{35}$$

If $\phi(\alpha_f) < \psi(\alpha_f)$*, the reverse direction in the above implications also hold, that is, they are actually if and only if statements.*

Proof First assume, for a contradiction, that $\mathrm{Q}_\tau^p[u(\varphi(f))] > \psi(\alpha_f)$. This means that there exists x such that $u(x) \geq \alpha_f$ and $\mathrm{Q}_\tau^p[u(\varphi(f))] > u(\varphi(x))$. But then

$$p\left(\{s \in S : u(f(s)) \leq u(x)\}\right) \geq \tau > p\left(\{s \in S : u(\varphi(f(s))) \leq u(\varphi(x))\}\right),$$

where the first inequality comes from $u(x) \geq \alpha_f$ and the second from the definition of $\mathrm{Q}_\tau^p[u(\varphi(f))]$ and the inequality $\mathrm{Q}_\tau^p[u(\varphi(f))] > u(\varphi(x))$. But this is an absurd, since $\{s \in S : u(f(s)) \leq u(x)\} \subset \{s \in S : u(\varphi(f(s))) \leq u(\varphi(x))\}$.

Next, we will show that $\mathrm{Q}_\tau^p[u(\varphi(f))] = \alpha_{\varphi(f)} \geq \phi(\alpha_f)$. For a contradiction, assume that $\alpha_{\varphi(f)} < \phi(\alpha_f)$. This means that there exists x such that $u(x) < \alpha_f$ and $\alpha_{\varphi(f)} < u(\varphi(x)) \leq \phi(\alpha_f)$. Let $\alpha = u(x)$. Then, by Lemma A.5, there exists x' such that $\alpha = u(x) < u(x') < \alpha_f$ and $\alpha_{\varphi(f)} < u(\varphi(x)) \leq u(\varphi(x')) \leq \alpha_{\varphi(f)}$, which is an absurd. Therefore, $\mathrm{Q}_\tau^p[u(\varphi(f))] = \alpha_{\varphi(f)} \geq \phi(\alpha_f)$.

To verify (34), assume that $p\left(\{s \in S : u(f(s)) < \alpha_f\}\right) \geq \tau$. By definition, for any $x \in \Upsilon$, $u(x) < \alpha_f$ implies $u(\varphi(x)) \leq \phi(\alpha_f)$. Therefore,

$$\tau \leq p\left(\{s \in S : u(f(s)) < \alpha_f\}\right) \leq p\left(\{s \in S : u(\varphi(f(s))) \leq \phi(\alpha_f)\}\right),$$

which implies $\mathrm{Q}_\tau^p[u(\varphi(f))] \leq \phi(\alpha_f)$, from which (34) follows.

To prove (35), assume that the inequality in its left holds and that $Q_\tau^p[u(\varphi(f))] = \alpha_{\varphi(f)} < \psi(\alpha_f)$. In this case,

$$\begin{aligned}\tau &> p\left(\{s \in S : u(f(s)) < \alpha_f\}\right)\\ &\geq p\left(\{s \in S : u(\varphi(f(s))) < \psi(\alpha_f)\}\right)\\ &\geq p\left(\{s \in S : u(\varphi(f(s))) \leq Q_\tau^p[u(\varphi(f))]\}\right)\\ &\geq \tau,\end{aligned}$$

where the second inequality comes from the definition of $\psi(\alpha)$, which guarantees that $u(\varphi(f(s))) < \psi(\alpha) \Rightarrow u(f(s)) < \alpha$; the third inequality comes from $Q_\tau^p[u(\varphi(f))] < \psi(\alpha_f)$ and the last from (32). This contradiction establishes (35).

Since the inequalities on the left in (34) and (35) are complementary, one of its conclusions must hold, that is, at least one of the inequalities (33) hold with equality. Finally, assume that $\phi(\alpha_f) < \psi(\alpha_f)$. To see that the reverse implication in (34) holds, consider two cases. If equality on the right hand side of (34) is false, the reverse implication holds trivially. If the equality is true, then the equality on the right hand side of (35) is false. By contraposition, the inequality on the left of (35) is also false. Therefore, the inequality on the left of (34) is true, which shows the reverse implication in (34). The proof for (35) is analogous. This concludes the proof of Proposition A.6. □

Finally, we restate and prove Proposition 3.10 as the following corollary.

Corollary A.7 *Assume that*

$$p\left(\{s \in S : u(f(s)) < \alpha_f\}\right) < \tau \text{ or } p\left(\{s \in S : u(g(s)) < \alpha_g\}\right) \geq \tau. \tag{36}$$

Then $\alpha_f \geq \alpha_g \Rightarrow \alpha_{\varphi(f)} \geq \alpha_{\varphi(g)}$.

Proof For a contradiction, assume that $\alpha_f \geq \alpha_g$ and $\alpha_{\varphi(f)} < \alpha_{\varphi(g)}$. By Lemma A.6, we have four cases to consider: (1) $\alpha_{\varphi(f)} = \psi(\alpha_f)$ and $\alpha_{\varphi(g)} = \psi(\alpha_g)$; (2) $\alpha_{\varphi(f)} = \phi(\alpha_f)$ and $\alpha_{\varphi(g)} = \phi(\alpha_g)$; (3) $\alpha_{\varphi(f)} = \psi(\alpha_f)$ and $\alpha_{\varphi(g)} = \phi(\alpha_g)$; and (4) $\alpha_{\varphi(f)} = \phi(\alpha_f)$ and $\alpha_{\varphi(g)} = \psi(\alpha_g)$.

The first two cases are immediate just using the monotonicity of ψ and ϕ established in Proposition A.4. For instance, $\phi(\alpha_g) > \phi(\alpha_f) \Rightarrow \alpha_g > \alpha_f$, which contradicts $\alpha_f \geq \alpha_g$. Case (3) also follows from monotonicity, since $\psi(\alpha_g) \geq \phi(\alpha_g) > \psi(\alpha_f) \Rightarrow \alpha_g > \alpha_f$. Thus, we need to consider only case (4), that is,

$$\psi(\alpha_g) = \inf\{u(\varphi(x)) : u(x) \geq \alpha_g\} > \sup\{u(\varphi(x)) : u(x) < \alpha_f\} = \phi(\alpha_f).$$

If there exists x such that $\alpha_f > u(x) \geq \alpha_g$, we would have

$$\psi(\alpha_g) = \inf\{u(\varphi(x)) : u(x) \geq \alpha_g\} \leq u(\varphi(x)) \leq \sup\{u(\varphi(x)) : u(x) < \alpha_f\} = \phi(\alpha_f),$$

contradicting the assumption. Therefore, $u(x) < \alpha_f \Rightarrow u(x) < \alpha_g$, $u(x) \geq \alpha_g \Rightarrow u(x) \geq \alpha_f$, and

$$\psi(\alpha_g) = \inf\{u(\varphi(x)) : u(x) \geq \alpha_g\} \geq \inf\{u(\varphi(x)) : u(x) \geq \alpha_f\} = \psi(\alpha_f).$$

Since $\alpha_f \geq \alpha_g \Rightarrow \psi(\alpha_f) \geq \psi(\alpha_g)$, we conclude that $\psi(\alpha_f) = \psi(\alpha_g)$. Similarly,

$$\phi(\alpha_f) = \sup\{u(\varphi(x)) : u(x) < \alpha_f\} \leq \sup\{u(\varphi(x)) : u(x) < \alpha_g\} = \phi(\alpha_g),$$

and $\alpha_f \geq \alpha_g \Rightarrow \phi(\alpha_f) \geq \phi(\alpha_g)$ imply $\phi(\alpha_f) = \phi(\alpha_g)$. Therefore,

$$\phi(\alpha_f) = \phi(\alpha_g) < \psi(\alpha_f) = \psi(\alpha_g).$$

This means that the last condition in Proposition A.6 are satisfied by both f and g. Since $\alpha_{\varphi(f)} = \phi(\alpha_f)$ and $\alpha_{\varphi(g)} = \psi(\alpha_g)$, Proposition A.6 implies that

$$p\left(\{s \in S : u(f(s)) < \alpha_f\}\right) \geq \tau > p\left(\{s \in S : u(g(s)) < \alpha_g\}\right),$$

but this contradicts (36). □

Proof of Theorem 3 As usual, it is easy to verify that the axioms are satisfied if the preference has the representation, that is, satisfies the recursive equation (18). Conversely, from Proposition 4.1, we know that if $\succcurlyeq$ satisfies D1–D7, then $\succcurlyeq$ admits a recursive representation (V, W, I) such that $W(c, x) = u(c) + b(c)x$, with $b(c) \in (0, 1)$. From Koopmans (1972) and A1, $b(c)$ is constant, that is, $b(c) = \beta \in (0, 1)$.

Let $\mathcal{F}$ denote, as before, the set of functions $f : S \to C$. Fix some $c^\infty = (c_0, c_1, c_2, \ldots) \in C^\infty$, define $\succcurlyeq^*$ on $\mathcal{F}$ by:

$$f \succcurlyeq^* g \iff (c_0, f(\cdot), c_2, \ldots) \succcurlyeq (c_0, g(\cdot), c_2, \ldots). \tag{37}$$

This preference is well defined and does not depend on $c^\infty \in C^\infty$. It is clear that $\succcurlyeq^*$ satisfies Q1–Q5. Therefore, by Theorem 2, there exists $p : \Sigma \to [0, 1]$ and $\tau \in (0, 1)$ such that

$$f \succcurlyeq^* g \iff \mathrm{Q}_\tau^p[u(f)] \geq \mathrm{Q}_\tau^p[u(g)] \tag{38}$$

where

$$\mathrm{Q}_\tau^p[u(f)] \equiv \inf\{\alpha : p(\{s \in S : u(f(s)) \geq \alpha\}) \leq \tau\}.$$

Since quantiles preferences are invariant to strictly increasing and continuous transformations, we can use in (38) the same $u : C \to [0, 1]$ provided by Proposition 4.1. By the definition of $\succcurlyeq^*$ and the recursive representation (V, W, I),

$$f \succcurlyeq^* g \iff I[u(f)] \geq I[u(g)].$$

Therefore, by (38), there exists a strictly increasing function $\xi : \mathbb{R} \to \mathbb{R}$ such that $I[u(f)] = \xi\left(\mathrm{Q}_\tau^p[u(f)]\right)$ for any $f \in \mathcal{F}$. Since $I[x] = x = \mathrm{Q}_\tau^p[x]$ for any $x \in \mathbb{R}$, ξ must be the identity. □

Proof of Proposition 5.2 See Manski (1988). □

Proof of Proposition 5.4 See Rostek (2010, Section 6.1) or Manski (1988, Section 5) for the equivalence of (1) and (2). The other two equivalences follow from Proposition 5.2. □

Proof of Proposition 5.5 For contradiction assume that there exists $f : S \to C$ such that $I^1[u(f)] > I^2[u(f)]$. Define $h = (h_0, f, f, \ldots)$. Pick $c \in C$ such that $I^1[u(f)] > u(c) > I^2[u(f)]$. Let $c^\infty = (c, c, \ldots)$. Then,

$$V^1(h) = u(h_0) + \beta I^1[V^1(h^1)] > V^1(c^\infty) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$= V^2(c^\infty) \geq V^2(h) = u(h_0) + \beta I^1[V^2(h^1)],$$

that is, $V^2(c^\infty) \geq V^2(h)$ but $V^1(h) > V^1(c^\infty)$, thus contradicting (19). Conversely, $I^1[\cdot] \leq I^2[\cdot]$ implies $V^1(h) \leq V^2(h)$ for any $h \in \mathcal{H}$. Thus, it cannot happen the negation of (19), that is, $V^1(h) > V^1(c^\infty) = V^2(c^\infty) \geq V^2(h)$. □

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